§ 10.2 / § 10.3 Adams page 581 - 590.

Vector

$\vec{u} = (x, y) \in \mathbb{R}^2 = V$ = vector space is 2-dimensional

$\vec{v} = (x, y, z) \in \mathbb{R}^3$

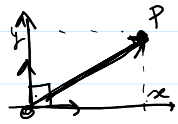
$\vec{w} = (x, y, z, t) \in \mathbb{R}^4$

$\vec{m} = (x_1, x_2, \dots x_n) \in \mathbb{R}^n = V$ is n-dimensional

$\vec{u} = (x, y) = (x, 0) + (0, y) = x(1, 0) + y(0, 1) = x\vec{i} + y\vec{j}$

$\{\vec{i}, \vec{j}\}$ is the standard Base of $\mathbb{R}^2 = V$.

$(0,1) = \vec{j}$



$0 \quad \vec{i} = (1, 0)$

$\overrightarrow{OP} = \vec{u} = P - O = (x, y) - (0, 0) = (x, y)$

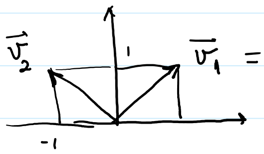
Def: Linear Combination:

$\vec{u}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$ if

$\exists\ \alpha_1, \alpha_2, \dots \alpha_k$ such that

$\vec{u} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_k \vec{v}_k$

$(-1, 1) = \vec{v}_2$



$\vec{v}_1 = (1, 1)$

$\{\vec{v}_1, \vec{v}_2\} = \{(1, 1), (-1, 1)\}$

? $\vec{i} = (1, 0)$ as a linear comb. of $\{\vec{v}_1, \vec{v}_2\}$?

Solution: We need to find constants α_1 & α_2 such that:

$$(1, 0) = \alpha_1 \overbrace{(1, 1)}^{v_1} + \alpha_2 \overbrace{(-1, 1)}^{v_2}$$

$$\begin{cases} 1\alpha_1 - 1\alpha_2 = 1 \\ 1\alpha_1 + 1\alpha_2 = 0 \Rightarrow \alpha_1 = -\alpha_2 \end{cases}$$

$-\alpha_2 - \alpha_2 = 1 \Rightarrow -2\alpha_2 = 1$

$\boxed{\alpha_2 = -1/2}$

$$\alpha_1 = -\left(-\tfrac{1}{2}\right) = +\tfrac{1}{2}$$

$$\boxed{(1,0) = \frac{1}{2}(1,1) - \frac{1}{2}(-1,1)}$$

$$(0,1) = \alpha(1,1) + \beta(-1,1)$$

$$\begin{cases} 1\alpha - 1\beta = 0 \Rightarrow \alpha = \beta \\ \alpha + \beta = 1 \quad \beta + \beta = 1 \Rightarrow 2\beta = 1 \quad \boxed{\beta = 1/2} \end{cases}$$

$$\boxed{\alpha = \beta = \frac{1}{2}}$$

$$(0,1) = \underbrace{\frac{1}{2}}_{\frac{1}{2}}(1,1) + \frac{1}{2}(-1,1)$$

$$\vec{u} = \vec{v} \iff (u_1, u_2 \dots u_n) = (v_1, \dots, v_n) \iff$$

$$u_1 = v_1$$
$$\vdots$$
$$u_n = v_n \quad \forall n \in \mathbb{N}, \quad V = \mathbb{R}^n.$$

$$\|\vec{v}\| = |\vec{v}| = \sqrt{v_1^2 + \dots v_n^2}, \quad \vec{v} \in \mathbb{R}^n, \quad n \in \mathbb{N}$$

Def 3 sida 581 Dot product

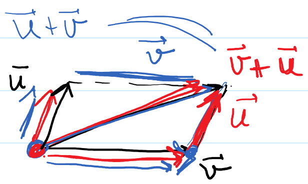
$\bar{u}, \vec{v} \in V = \mathbb{R}^n$ $\quad \bar{u} = (u_1, \dots, u_n)$, $\bar{v} = (v_1, \dots, v_n)$ $\quad n \in \mathbb{N}.$

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{k=1}^{n} u_k \cdot v_k}$$

$n = 2$ or $n = 3$ $\quad (\mathbb{R}^2, \mathbb{R}^3)$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| . \cos\theta \qquad \theta = \measuredangle(\vec{u}, \vec{v})$$

angle

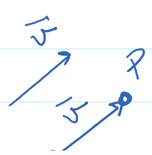


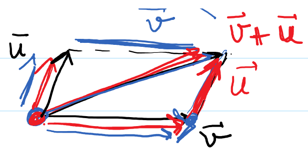
$$\bar{u} + \bar{v} = \vec{v} + \vec{u}$$

$\forall$ = for all values

A = A big!

$\forall n \in \mathbb{R}$





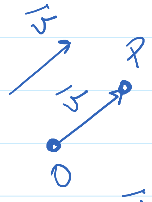
angle

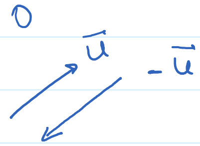
$$\vec u + \vec v = \vec v + \vec u$$

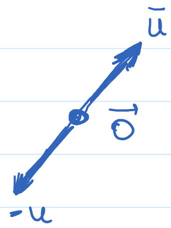
$\forall$ = for all values

A = A big!

$\forall n \in \mathbb{R}$

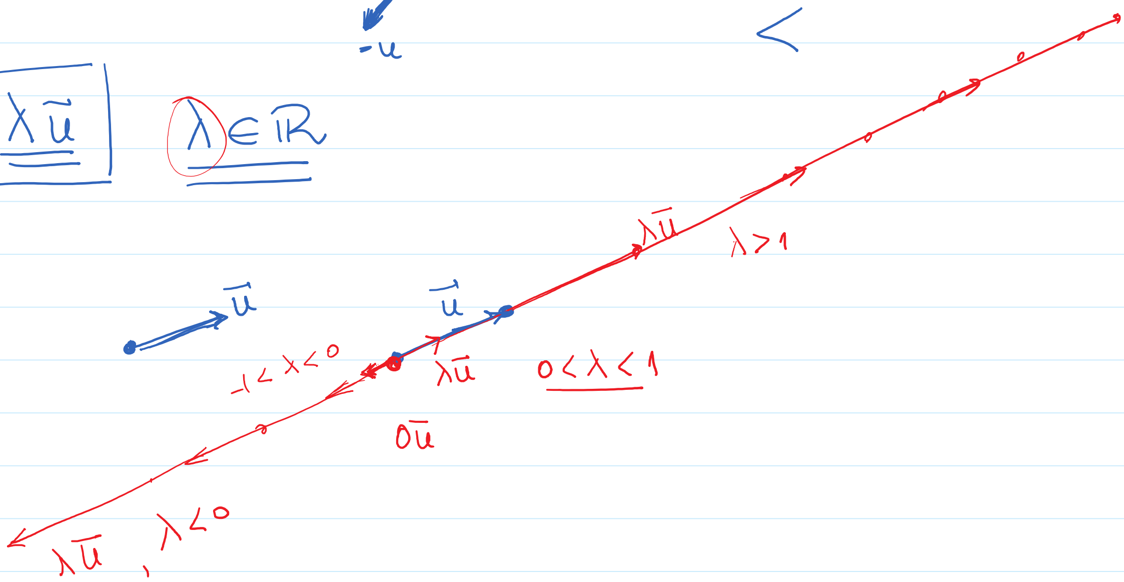






$$\vec u + (-\vec u) = \vec 0$$

$\boxed{\lambda \vec u}$ $\lambda \in \mathbb{R}$



Def 4. Scalar & Vectorial projections

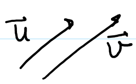
1) The scalar projection s of a vector $\vec u$ in the direction of a vector $\vec v$ $(\vec v \neq \vec 0)$ is the dot product of $\vec u$ with a unit vector in the direction of $\vec v$.

$\vec v \neq \vec 0$ $\quad \|\vec v\| \neq 0$ $\quad \hat v = \dfrac{\vec v}{\|\vec v\|}$ ← obs $\hat v$ is a vector // $\vec v$
$|\hat v| = 1$.

// ≡ parallel

$\hat v = \dfrac{1}{\|\vec v\|}\vec v$ $\quad (\hat v \, // \, \vec v)$

$\vec u \, // \, \vec v \Leftrightarrow \vec u = \lambda \vec v$



$$\|\hat v\| = \left\| \frac{1}{\|\vec v\|} \vec v \right\| = \frac{1}{\cancel{\|\vec v\|}} \cancel{\|\vec v\|} = 1$$

$$s = \vec u \cdot \frac{\vec v}{|\vec v|} = \frac{|\vec u|\cancel{|\vec v|}.\cos\theta}{\cancel{|\vec v|}} = |\vec u|.\cos\theta$$

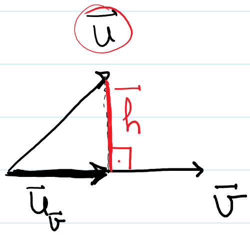
$\theta = \measuredangle(\vec u, \vec v)$
angle between $\vec u$ & $\vec v$.

2) The <u>vector projection</u> $\vec{u}_{\vec{v}}$ of $\vec{u}$ in the direction of $\vec{v}$ is the scalar multiple (of a unit vector $\vec{v}$) in the direction of $\vec{v}$

$$\vec{u}_{\vec{v}} = \frac{(\vec{u}\cdot\vec{v})}{(\vec{v}\cdot\vec{v})}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\left(\frac{\vec{v}}{|\vec{v}|}\right) =$$

$$= \frac{\vec{u}\cdot v}{|v|}\hat{v}$$

$\hat{v}$ ⟵ unit vector $/\!/ \ \vec{v}$ parallel to $\vec{v}$.



$\vec{u}_{\vec{v}} = \lambda\vec{v}$

$\boxed{\vec{u}_{\vec{v}}}$ = { projection of $\vec{u}$ in the $\vec{v}$ direction

$$\boxed{\lambda\vec{v} + \vec{h} = \vec{u}}$$

$\lambda = ?$ what is this <u>value</u>

$$\lambda\vec{v}\cdot\vec{v} + \vec{h}\cdot\vec{v} = \vec{u}\cdot\vec{v}$$

$h \perp \vec{v} \Rightarrow h\cdot\vec{v} = 0$

$\perp = \measuredangle(u,v) = 90^\circ$ { perpendicular, orthogonal

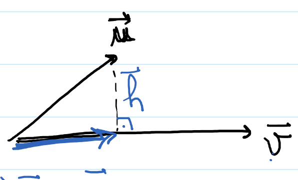
$$\lambda\vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{v}$$

$$\lambda = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}$$

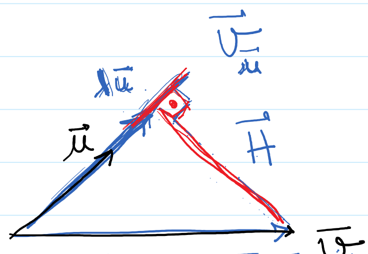
$$\lambda\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} \cdot \vec{v} = \vec{u}_{\vec{v}}.$$

n°   vector

<u>Obs</u>: $\vec{u}_{\vec{v}} \neq \vec{v}_{\vec{u}}$



$\lambda\vec{v} = \vec{u}_{\vec{v}}$



$$\lambda\vec{u} + \vec{H} = \vec{v}$$

$\lambda\vec{v} = \vec{u}_{\vec{v}}$

$$\vec{u}_{\vec{v}} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

$$\boxed{\lambda\vec{u} + \vec{H} = \vec{v}}$$

$$\lambda\,\vec{u}\cdot\vec{u} + \cancel{\underbrace{\vec{H}\cdot\vec{u}}_{\text{perpendicular}}} = \vec{v}\cdot\vec{u}$$

$\theta.\ \measuredangle(H, \vec{u}) = 90^\circ \Rightarrow \cos 90^\circ = 0$

$$\lambda = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$$

$$\vec{v}_{\vec{u}} = \left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

$\vec{i} = \vec{u} = (1,0) \qquad \vec{v} = (1,1)$

$$\vec{u}_{\vec{v}} = \lambda\vec{v} \qquad \lambda = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} = \frac{(1,0)\cdot(1,1)}{(1,1)\cdot(1,1)} = \frac{1+0}{1+1} = \frac{1}{2}$$

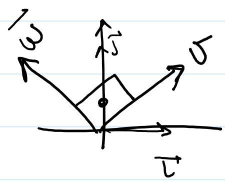
$\vec{i} = \vec{u} = (1,0) \qquad \vec{w} = (-1,1)$

$$\vec{u}_{\vec{w}} = \lambda\vec{w} \qquad \lambda = \frac{\vec{u}\cdot\vec{w}}{\vec{w}\cdot\vec{w}} = \frac{(1,0)\cdot(-1,1)}{(-1,1)\cdot(-1,1)} = \frac{-1+0}{+1+1} = \frac{-1}{2}$$

$$\vec{i} = (1,0) = \frac{1}{2}\vec{v} + \left(-\frac{1}{2}\right)\vec{w}$$

$$= \frac{1}{2}(1,1) + \left(-\frac{1}{2}\right)(-1,1)$$

$$\vec{v}\cdot\vec{w} = (1,1)\cdot(-1,1) = -1+1 = \underline{\underline{0}} \Longrightarrow \underline{\vec{v}\perp\vec{w}}$$



$\mathbb{R}^2$

Remark

$\vec{u}\cdot\vec{v}$ works fine in all dimensions

§ 10.3 page 585

The Cross Product in $\mathbb{R}^3$

The Cross Product in $\mathbb{R}^3$

To remember: what we have so far:

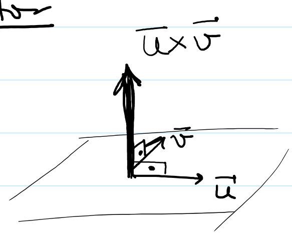
1) $\lambda\vec{u}$ = scalar product (is a vector parallel to $\vec{u}$)

2) $\vec{u}\bullet\vec{v}$ = dot product (is a number)

3) $\vec{u}\times\vec{v} \equiv$ CROSS product which produces another **vector**!

Definition 5: For any $\vec{u},\vec{v}\in\mathbb{R}^3$

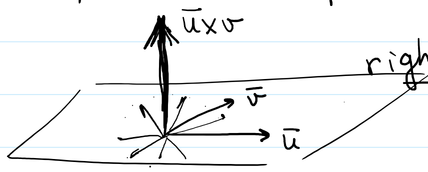
The cross product $\vec{u}\times\vec{v}$ is the unique vector satisfying the following 3-conditions:



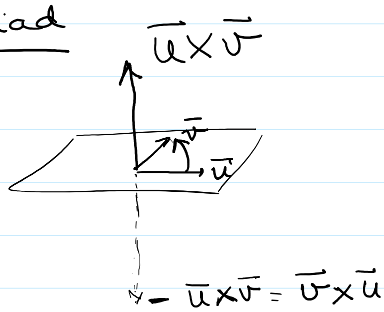
1) $(\vec{u}\times\vec{v})\bullet\vec{u} = 0 \Rightarrow (\vec{u}\times\vec{v})\perp\vec{u}$
   $(\vec{u}\times\vec{v})\bullet\vec{v} = 0 \Rightarrow (\vec{u}\times\vec{v})\perp\vec{v}$

2) $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$, $\theta = \measuredangle(\vec{u},\vec{v})$ = angle between $\vec{u}$ & $\vec{v}$.

3) $\{\vec{u},\vec{v},\vec{u}\times\vec{v}\}$ forms a right-handed Triad



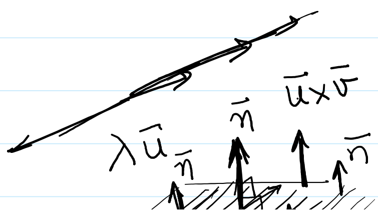
obs:

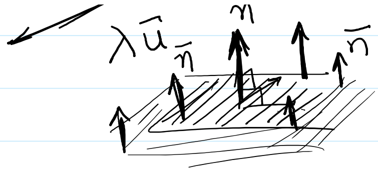


If $\vec{u}/\!/\vec{v}$ $(\vec{u}=\lambda\vec{v})$

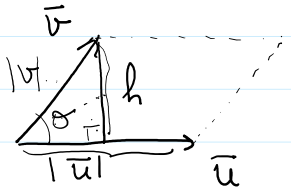
$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ $\theta = \measuredangle(\vec{u},\vec{v}) = 0$

$|\vec{u}\times\vec{v}| = 0 \Rightarrow \boxed{\vec{u}\times\vec{v} = \vec{0}}$





2) $|\bar{u} \times \bar{v}| = \underline{|\bar{u}||\bar{v}| \sin\theta}$



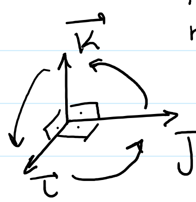
parallelogram formed by $\bar{u}$ & $\bar{v}$.

$$\text{Area} = \text{base} \times \underbrace{\text{height}}_{h}$$
$$= |\bar{u}| \qquad h$$

$$\sin\theta = \frac{h}{|v|} \Rightarrow \boxed{h = |v| \sin\theta} \qquad \theta = \measuredangle(\bar{u}, \bar{v})$$

$$\text{Area} = |\bar{u}| \cdot |\bar{v}| \sin\theta = |\bar{u} \times \bar{v}|$$

$\{\bar{i}, \bar{j}, \bar{k}\}$ right-Handed triad in $\mathbb{R}^3$.



Theorem 2 page 585

$$\bar{u} = (u_1, u_2, u_3) = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$$
$$\bar{v} = (v_1, v_2, v_3) = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

Then

$$\bar{u} \times \bar{v} = \underbrace{(u_2 v_3 - u_3 v_2, -u_1 v_3 + u_3 v_1, u_1 v_2 - u_2 v_1)} = \bar{w}$$

<u>Proof</u> = Take a look in the Book.

$\bar{w} \cdot \bar{u} = 0 \quad \bar{w} \cdot \bar{v} = 0 \quad \ldots$ and the 3 properties are satisfied

---

<u>Define</u>: <u>Determinant</u>

1) $M_{1\times1} = [a] \qquad \underbrace{|M_{1\times1}|}$ = determinant of this matrix

$$= a$$

2) $M_{2\times2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad |M_{2\times2}| = a.d - b.c$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$= (-1)^{1+1} a\,|m_{22}| + (-1)^{1+2} b\,|m_{21}|$$

$$= 1.ad - bc$$

3) $M_{3\times3} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + dhc + gbf$

$$- (gec + hfa + idb)$$