

In this extra material, I would like to finally answer the question about the types of equations of straight lines when we are in $\Pi = \mathbb{R}^2 = \Pi_{xy}$ or in $\mathbb{R}^3$.

So: Given a known Point $P_0 = (x_0, y_0, z_0)$ & a direction $\vec{v} = (a, b, c)$ we can construct the equations of the straight line r assuming that:

1) Any Point X that belongs to the line r will form a oriented segment $\overrightarrow{P_0X}$ parallel to $\vec{v}$.

So

$$\boxed{\overrightarrow{P_0X} = \lambda \vec{v}} \Leftrightarrow \boxed{X - P_0 = \lambda \vec{v}} \Leftrightarrow \boxed{X = P_0 + \lambda \vec{v}}$$

vector parametric eq of r .

$$2) X = P_0 + \lambda \vec{v} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Leftrightarrow$$

$$\begin{cases} x = x_0 + \lambda a \\ y = y_0 + \lambda b \\ z = z_0 + \lambda c \end{cases}$$

$\lambda \in \mathbb{R} \quad \lambda \in (-\infty, +\infty)$

These are the
Scalar parametric
equations of r .

3) And isolating λ on (2) we have:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad \text{the standard form of the equations of } r.$$

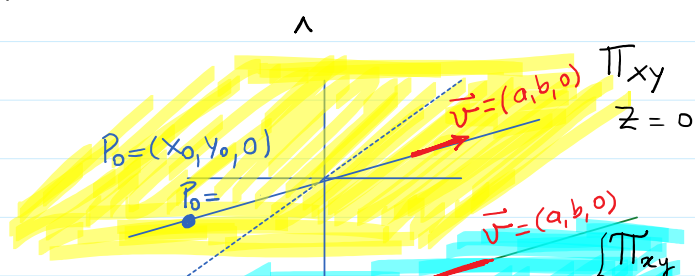
$$(a \neq 0, b \neq 0, c \neq 0)$$

But now, let's think in the case when $\boxed{c=0}$

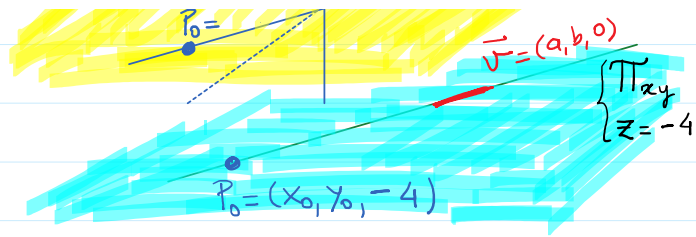
So. the equation of r is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

$$\begin{cases} x = x_0 + \lambda a \\ y = y_0 + \lambda b \\ z = z_0 \end{cases}$$



$$\begin{cases} y = y_0 + \lambda b \\ z = z_0 \end{cases}$$



And Now, we have to modify the

standard form of the line equation since $c=0$

$$\begin{cases} \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \\ z = z_0 \end{cases}$$

$$\Pi_{xy}: \begin{cases} \frac{x-x_0}{a} = \frac{y-y_0}{b} \\ z = 0 \end{cases}$$

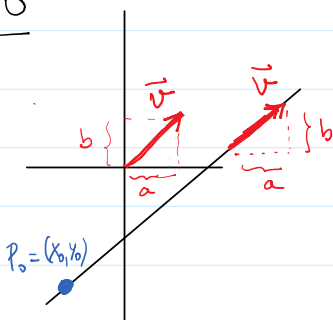
This is the original plane we associated to $\mathbb{R}^2$

$$\Pi_{xy}: \begin{cases} \frac{x-x_0}{a} = \frac{y-y_0}{b} \\ z = -4 \end{cases}$$

This is a parallel plane to Π_{xy} , but passing through the point $P=(0,0,-4)$

Now let's look closer to this standard form of the equation of r in $\mathbb{R}^2 = \Pi_{xy}, z=0$.

$z=0$



$$\frac{\Delta y}{\Delta x} = \frac{b}{a} = \alpha$$

If we write the eq of line as we are use to, we have:

$$y - y_0 = \alpha (x - x_0)$$

$$y - y_0 = \frac{b}{a} (x - x_0)$$

$$a(y - y_0) = b(x - x_0)$$

$\Rightarrow$

$$\frac{(y - y_0)}{b} = \frac{(x - x_0)}{a}$$

$$\begin{cases} \frac{x-x_0}{a} = \lambda \\ \frac{y-y_0}{b} = \lambda \\ x-x_0 = \lambda a \\ y-y_0 = \lambda b \end{cases} \Rightarrow$$

$$\begin{cases} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \end{pmatrix} \\ z = 0 \end{cases}$$

$$\frac{b}{a} = \frac{a}{b} \quad \left\{ \begin{array}{l} (y) = (y_0) + \lambda (b) \\ z = 0 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \end{array} \right.$$

with $z=0$ or $z=-4$ or $z=k$
 of any height we choose to
 form a plane parallel to Π_{xy}
 but passing through $P_0 = (x_0, y_0, k)$

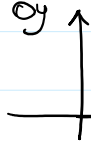
$$S_0 \quad \frac{x-x_0}{a} = \frac{y-y_0}{b} \Rightarrow \frac{b}{a} (x-x_0) = y-y_0 \Rightarrow$$

$$y = y_0 + \left(\frac{b}{a} \right) (x-x_0)$$

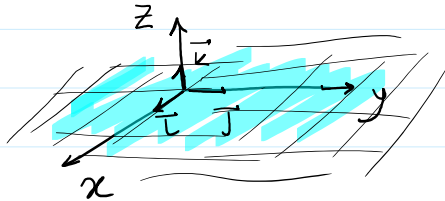
slope of our $\vec{v} \equiv$ direction
 vector of r .

• Now let's think again about the equation of the axis

$\vec{Oy}$
 $x=0$
 y free



x free
 $y=0$ is the equation of $\vec{Ox}$



But we have to remember
 that $z=0$, since we are in
 $\Pi_{xy} = \mathbb{R}^2$.

To obtain the equation of $\vec{Ox}$,

we want $P_0 = (0,0,0)$ point of our line r .

$\vec{t} = (1,0,0)$ direction vector of our line.

$$\vec{XP_0} = \lambda \vec{t} \Rightarrow X - P_0 = \lambda \vec{t} \Rightarrow X = P_0 + \lambda \vec{t}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 + \lambda \\ 0 + 0 \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$$

$x = \lambda \leftarrow x$ is free to assume any value in $\mathbb{R}$
 $y=0$ fixed in $\mathbb{R}^2$
 $z=0$ fixed because we are looking $\mathbb{R}^2$ inside $\mathbb{R}^3$

$\boxed{y=0}$ fixed in $\mathbb{R}^2$
 $\boxed{z=0}$ fixed because we are seeing $\mathbb{R}^2$ inside $\mathbb{R}^3$.

Analogously, we have

$$\overline{Oy}: P = (0,0,0) \quad \vec{v} = \vec{j} = (0,1,0)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \cdot 1 \\ 0 \end{pmatrix}$$

So $x=0$ is the equation of $\overline{Oy}$.
 $y = \text{free}$
 $z=0$

& the standard form in $\mathbb{R}^3$ is

$$x=0 \quad z=0 \quad \boxed{\frac{y-0}{1} = \lambda} \quad \begin{pmatrix} y-0 = 1 \cdot \lambda \\ y = \lambda \end{pmatrix}$$

I hope now I have answered the question in a more detailed way. 😊

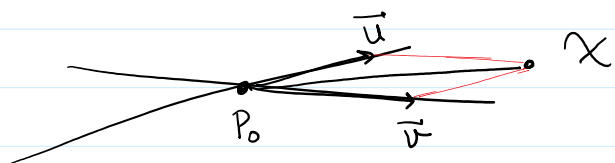
II. About Π and its vectorial equation.

⊗ Obs 1. Two vectors not parallel define a plane Π

Obs 2 To be able to localize this plane in space we need to know a point $P_0 = (x_0, y_0, z_0)$ that belongs to Π .

Otherwise, we have a family of infinite planes (all parallel to each other)

$$\Pi: P_0 = (x_0, y_0, z_0) \quad \vec{u} = (a, b, c) \quad \vec{v} = (m, n, p)$$



Every point $X \in \Pi$, forms a vector $\overrightarrow{P_0X}$

which is a linear combination of $\vec{u}$ & $\vec{v}$.

$$\vec{P_0X} = \alpha \vec{u} + \beta \vec{v}$$

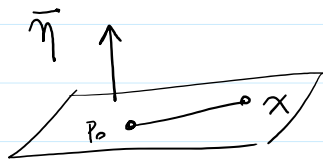
$$X - P_0 = \alpha \vec{u} + \beta \vec{v}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}}_{\text{straight line } r} + \alpha \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \beta \underbrace{\begin{pmatrix} m \\ n \\ p \end{pmatrix}}_{\text{straight line } s}$$

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = r \cap s = \text{intersection of two lines in } \mathbb{R}^3$$

So, this is another way to think about planes in $\mathbb{R}^3$.

III Π & the point-normal equation. $\begin{cases} P_0 \text{ given} \\ \vec{n} \text{ given} \end{cases}$



$$\vec{P_0X} \perp \vec{n}$$

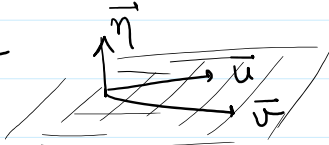
$$(x - x_0, y - y_0, z - z_0) \perp (a, b, c) \Rightarrow$$

$$(x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0 \Rightarrow$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = (ax_0 + by_0 + cz_0) = d$$

- Here (x, y, z) are the coordinates of the generic point X which belongs to Π .

- $\vec{n} = (a, b, c) = \vec{u} \times \vec{v}$  any two vectors of Π , that are NOT parallel.

Obs. $2x + 4y + 7z = -1 \quad \Pi \subseteq \mathbb{R}^3$

To find points that belong to Π ,

we choose 2 variables, and obtain the third satisfying the equation:

For example:

$P_1 = (0, 0, z)$ z being such that
 $2(0) + 4(0) + 7z = -1 \Rightarrow z = -\frac{1}{7}$

$P_1 = (0, 0, -\frac{1}{7})$

$P_2 = (x, 0, 0)$ x being such that
 $2x + 4(0) + 7(0) = -1 \Rightarrow x = -\frac{1}{2}$

So $P_2 = (-\frac{1}{2}, 0, 0)$

Och $P_3 = (1, y, 1)$, y being such that

$$2(1) + 4y + 7(1) = -1$$

$P_3 = (1, -\frac{5}{2}, 1)$ $4y = -1 - 2 - 7 \Rightarrow y = -\frac{10}{4}$

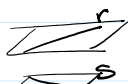
$$y = -\frac{5}{2}$$

So P_1, P_2 & P_3 are 3 points in $\mathbb{R}^3$.

So $\overrightarrow{P_1P_2}$ & $\overrightarrow{P_1P_3}$ for example can generate 2 vectors (not parallel) in Π .

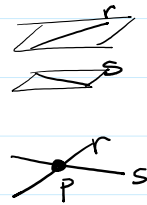
So $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \lambda \vec{n} = \lambda(2, 4, 7)$

Intersections

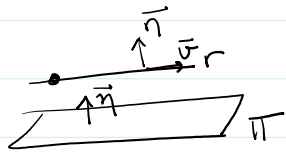


Intersections

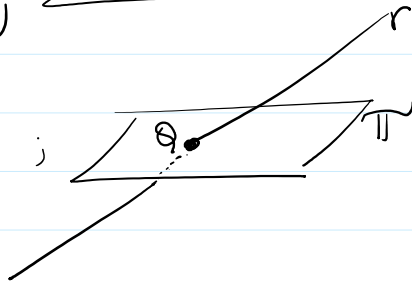
$$1) r \cap s = \begin{cases} \emptyset \\ P \end{cases}$$



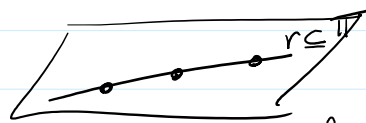
$$2) r \cap \Pi = \begin{cases} \emptyset, \\ P, \\ r, \text{ if } r \subseteq \Pi \end{cases}$$



$\therefore$ system without solution



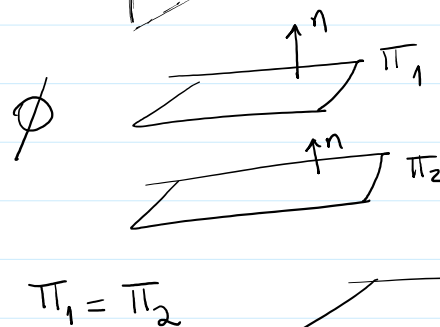
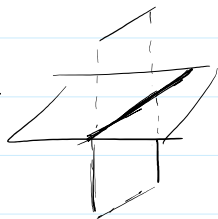
system with unique solution



system with infinite solution.

This mean there is one free parameter to express the solution.

$$3) \Pi_1 \cap \Pi_2 = \begin{cases} r = \\ \emptyset \end{cases}$$



Parallel Π_1 & Π_2

$\Pi_1 = \Pi_2$



the equation are multiple

Example 6 page 597

Find the intersection between $\Pi_1: x + y - z = 0$
& $\Pi_2: y + 2z = 6$.

& $\pi_2: y + 2z = 6$.

$$\begin{cases} \text{Eq (1)} & x + y - z = 0 \\ \text{Eq (2)} & 0x + 1y + 2z = 6 \end{cases} \Rightarrow y = 6 - 2z$$

I will substitute this in eq (1)

$$x + (6 - 2z) - z = 0$$

$$x + 6 - 3z = 0 \Rightarrow x = -6 + 3z$$

Now we observe that both x & y depend on z .

$$x = -6 + 3z$$

$$y = 6 - 2z$$

$$z = 0 + 1z$$

$$\boxed{z = z = \lambda} \text{ because } 1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ 0 \end{pmatrix} + \begin{pmatrix} 3\lambda \\ -2\lambda \\ 1\lambda \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

↑

point of r

direction vector of r

general point of r

$$r = \pi_1 \cap \pi_2$$

$$\frac{x+6}{3} = \frac{y-6}{-2} = \frac{z-0}{1}$$

$$\boxed{\frac{x+6}{3} = \frac{6-y}{2} = z}$$