Föreläsning 6.2 **Mål**: { • Determinant & crossproduct
• Equation of line
• Equation of Plane
• normal to a plane.

Determinant

1) $M_{1\times 1} = [a_{11}]$ $\quad |M_{11}| = a_{11}$ $\quad$ determinant of $M = |M|$

2) $M_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad |M_{22}| = ad - bc$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad = (-1)^{1+1} \cdot a \cdot |m_{22}| + \underbrace{(-1)^{1+2}} b\,|m_{21}| =$$

$$= a.d - bc$$

3) $$M_{3\times 3} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + dhc + gbf$$

$$-(gec + hfa + idb)$$

$$= (-1)^{1+1} a \begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)^{1+2} b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + (-1)^{1+3} c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - gf) + c(dh - eg)$$

$$= aei + bgf + cdh$$

$$-(afh + bdi + ceg)$$

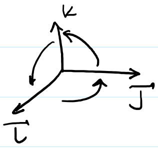
Now, we can think about $\vec{u} \times \vec{v}$ as being computed using determinants:

Page <u>585</u>:

1) Consider $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$ standard basis of $\mathbb{R}^3$.

so that $\vec{\imath} \times \vec{\jmath} = \vec{k}$, $\quad \vec{\jmath} \times \vec{k} = \vec{\imath}$

and $\vec{k} \times \vec{\imath} = \vec{\jmath}$ (right-Handed Triad)



2) $\vec{u} = (u_1, u_2, u_3) = u_1\vec{\imath} + u_2\vec{\jmath} + u_3\vec{k}$ (coordinates of $\vec{u}$ & $\vec{v}$
$\vec{v} = (v_1, v_2, v_3) = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$ with respect to the standard Basis)

So, $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \end{vmatrix}$

So, $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} =$

$$= (-1)^{1+1}\vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + (-1)^{1+2}\vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + (-1)^{1+3}\vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= +\vec{i}\,(u_2 \cdot v_3 - u_3 v_2) - \vec{j}\,(u_1 v_3 - u_3 v_1) + \vec{k}\,(u_1 v_2 - v_1 u_2)$$

$$= (u_2 v_3 - u_3 v_2)\vec{i} + (u_3 v_1 - u_1 v_3)\vec{j} + (u_1 v_2 - v_1 u_2)\vec{k} =$$

$$= (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - v_1 u_2)$$

Properties of $\vec{u} \times \vec{v}$ seen through properties of Determinant.
Pages 587 X 589

1) $\vec{u} \times \vec{u} = \vec{0}$ since $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ a & b & c \end{vmatrix} = \vec{i}\underbrace{\begin{vmatrix} b & c \\ b & c \end{vmatrix}}_{=0} - \vec{j}\underbrace{\begin{vmatrix} a & c \\ a & c \end{vmatrix}}_{=0} + \vec{k}\underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{=0}$

$$= 0\vec{i} + \vec{0}\vec{j} + \vec{0}\vec{k}$$

$$= (0,0,0) = \vec{0}.$$

2) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ since permutation of 2 lines in the matrix causes a change of the sign of the determinant

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ e & f & g \end{vmatrix} = \vec{i}\begin{vmatrix} b & c \\ f & g \end{vmatrix} - \vec{j}\begin{vmatrix} a & c \\ e & g \end{vmatrix} + \vec{k}\begin{vmatrix} a & b \\ e & f \end{vmatrix}$$

$$\vec{i}\,(\underline{bg - cf}) - \vec{j}\,(ag - ec) + \vec{k}\,(\underbrace{af - eb})$$

$$\vec{v}\times\vec{u} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e & f & g \\ a & b & c \end{vmatrix} = \vec{i}\begin{vmatrix} f & g \\ b & c \end{vmatrix} - \vec{j}\begin{vmatrix} e & g \\ a & c \end{vmatrix} + \vec{k}\begin{vmatrix} e & f \\ a & b \end{vmatrix} =$$

$$= i\,(fc - gb) - j\,(ec - ag) + \vec{k}\,(eb - af)$$

$$-\,(gb - fc) \quad -\vec{j}\,(-1)(ag - ec) \quad + \vec{k}\,(-1)(af - eb)$$

opposite signs

$$-(gb-fc) \quad -\vec{j}(-1)(ag-ec) \quad +\vec{k}\,(-1)(af-eb)$$

3) $\vec{u}\times(v+w) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$

$\phi \quad \vec{u}=(u_1,u_2,u_3)$

a) $\vec{v}+\vec{w} = (x,y,z) + (l,m,n) = (x+l,\, y+m,\, z+n)$

b) $\vec{u}\times(\vec{v}+\vec{w}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ x+l & y+m & z+n \end{vmatrix} =$

$$= \vec{i}\begin{vmatrix} u_2 & u_3 \\ y+m & z+n \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ x+l & z+n \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ x+l & y+m \end{vmatrix}$$

$$= i\left(u_2(z+n) - u_3(y+m)\right)$$

$$= i\left(u_2 z + u_2 n - u_3 y - u_3 m\right)$$

$$= i\left(u_2 z - u_3 y + u_2 n - u_3 m\right)$$

analogous

$$= i\begin{vmatrix} u_2 & u_3 \\ y & z \end{vmatrix} + i\begin{vmatrix} u_2 & u_3 \\ m & n \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ x & z \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ l & n \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ x & y \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ l & m \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ — & \vec{u} & — \\ \vec{v} & — & \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ — & \vec{u} & — \\ — & \vec{w} & — \end{vmatrix}$$

← coordinates of $\vec{u}$

← coordinates of $\vec{w}$.

4) $(\lambda\vec{u})\times\vec{v} = (\vec{u})\times(\lambda\vec{v}) = \lambda(\vec{u}\times\vec{v})$

$\vec{u}=(a,b,c) \qquad \lambda\vec{u} = (\lambda a, \lambda b, \lambda c$

$\vec{v}=(x,y,z)$

$(\lambda\vec{u})\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \lambda a & \lambda b & \lambda c \\ x & y & z \end{vmatrix} =$

$$\vec{i}\begin{vmatrix} \lambda b & \lambda c \\ y & z \end{vmatrix} - \vec{j}\begin{vmatrix} \lambda a & \lambda c \\ x & z \end{vmatrix} + \vec{k}\begin{vmatrix} \lambda a & \lambda b \\ x & y \end{vmatrix} =$$

$\vec{i}(\lambda bz - \lambda cy)$

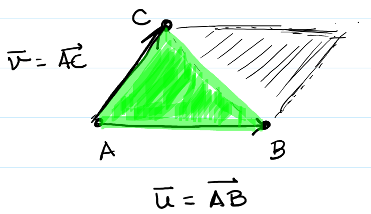
This because the property of determinant pag 589

$\lambda\vec{i}(bz-cy) \quad -\lambda\vec{j}(az-cx) \quad +\lambda\vec{k}(ay-bx)$

$$= \lambda \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ x & y & z \end{vmatrix} = \lambda (\vec{u} \times \vec{v})$$

Example 3 page 589

Find the area of the triangle with vertices given by the point in $\mathbb{R}^3$:
$A = (1,1,0)$ $B = (3,0,2)$ & $C = (0,-1,1)$



Area of the $\triangle$ $= \frac{1}{2}$ Area $\square$

$\vec{u} = \vec{AB} = B - A = (3-1, 0-1, 2-0) = (2,-1,2)$

$\vec{v} = \vec{AC} = C - A = (0,-1,1) - (1,1,0) = (-1,-2,1)$

$$\text{Area}_{\square} = \|\vec{u} \times \vec{v}\| = \left\| \underbrace{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ -1 & -2 & 1 \end{vmatrix}}_{} \right\| =$$

1st) compute $\vec{u} \times \vec{v}$

2nd) compute size of $\vec{u} \times \vec{v}$.

3°.) $A_{\triangle} = \frac{1}{2} A_{\square}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ -1 & -2 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ -1 & -2 \end{vmatrix} =$$

$$\vec{i}(-1+4) - \vec{j}(2+2) + \vec{k}(-4-1) =$$

$$i(3) - j(4) + \vec{k}(-5)$$

$\vec{u} \times \vec{v} = (3, -4, -5)$

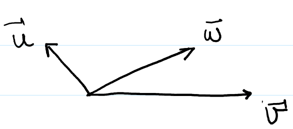
$$\|\vec{u} \times \vec{v}\| = \sqrt{(3)^2 + (-4)^2 + (-5)^2} = \sqrt{9 + 16 + 25} = \sqrt{50} = \sqrt{25 \cdot 2} =$$

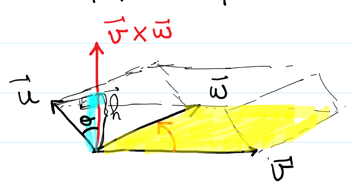
$$\|\vec{u} \times \vec{v}\| = 5\sqrt{2}$$

Thus $\text{Area}_{\triangle} = \frac{1}{2} \|\vec{u} \times \vec{v}\| = \frac{5}{2}\sqrt{2}$

Example 4 Page 590

Find the volume of the parallelepiped spanned by the vectors $\vec{u}, \vec{v}, \vec{w}$





To compute the volume V = (Area of base) × (height)

To compute the volume $V = \underbrace{(\text{Area of base})}_{|\vec{v}\times\vec{w}|} \times \underbrace{(\text{height})}_{|\vec{h}|}$

$$\|\vec{h}\| = \|\text{proj}_{\vec{v}\times\vec{w}}\,\vec{u}\| = \|\vec{u}_{\vec{u}\times\vec{w}}\| = \left\| \overbrace{\frac{\vec{u}\cdot(\vec{u}\times\vec{w})}{\|\vec{u}\times\vec{w}\|^2}}^{\lambda} (\vec{v}\times\vec{w}) \right\| = |\lambda|\,\|\vec{v}\times\vec{w}\|$$

$$= \frac{|\vec{u}\cdot(\vec{v}\times\vec{w})|}{\|\vec{v}\times\cancel{\vec{w}\|^2}} \cdot \|\vec{v}\cancel{\times\vec{w}\|}$$

And now:

$$|\vec{u}\cdot(\vec{v}\times\vec{w})| = \|\vec{u}\|\,\|\vec{v}\times\vec{w}\|\,|\cos\theta| \qquad \text{with } \theta = \measuredangle(\vec{u},(\vec{v}\times\vec{w})) \quad \left\{ \begin{array}{l} \text{angle between} \\ \vec{u} \,\&\, \vec{v}\times\vec{w}. \end{array} \right.$$

$$\|\vec{h}\| = \frac{|\vec{u}|\,\|\vec{v}\times\vec{w}\|\,|\cos\theta|}{\|\vec{v}\times\vec{w}\|}$$

$$\text{And Volume} = \overbrace{\frac{\|\vec{u}\|\cdot\|\vec{v}\times\vec{w}\|\cdot|\cos\theta|}{\|\vec{v}\times\cancel{\vec{w}\|}}}^{\|\vec{h}\|} \cdot \overbrace{\|\vec{v}\cancel{\times\vec{w}}\|}^{\text{Area}}$$

$$= |\vec{u}\cdot\vec{v}\times\vec{w}| \equiv \text{This is called the Scalar Triple product}$$

Definition 6 page 590.

Considering now the coordinates of these three vectors:

$\vec{u} = (u_1, u_2, u_3)$; $\vec{v} = (v_1, v_2, v_3)$ $\vec{w} = (w_1, w_2, w_3)$

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \vec{i}\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \vec{j}\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \vec{k}\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

So

$$\underbrace{\vec{u}\cdot(\vec{v}\times\vec{w})}_{\text{The Scalar triple product}} = u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\cdot\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

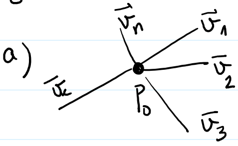
$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Can also be computed with the help of the determinant of a matrix having its coefficients given by the vectors coordinates.

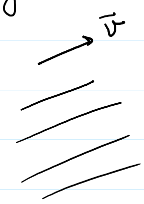
§ 10.4 page 593

Equation of line r $\begin{cases} P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3 \quad P_0 \in r \\ \vec{v} = \text{vector containing the direction of } r. \end{cases}$

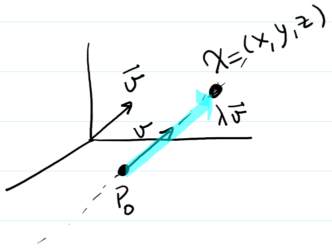
just one of these 2 information is not enough to define the line.

a) 

to one point passes many lines.

b) 

given 1 direction, there are infinite parallel lines



A point $X = (x, y, z)$ is point of the line r if

← Obs X is a generic point that belongs to the line r. So the coordinates of X MUST satisfy the equation of r.

* the oriented segment $\overline{P_0X}$ is parallel to $\vec{v}$.

$\therefore \overline{P_0X} = \lambda\vec{v}$.

* or there exists a constant value $\lambda \in \mathbb{R}$, such that $\overrightarrow{P_0X}$ is linear combination of $\vec{v}$.

$$\boxed{\overrightarrow{P_0X} = \lambda\vec{v}}$$

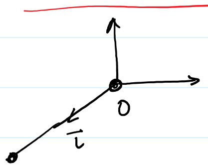
In any case, we say that $\vec{v}$ = direction vector of the line r.
And the line equation can be written in the following ways:

a) $P_0X = \lambda\vec{v} \Rightarrow X - P_0 = \lambda\vec{v} \Rightarrow \boxed{X = P_0 + \lambda\vec{v}}$

This is called the Vector parametric equation of the straight line

page 596 $\boxed{r = r_0 + t\vec{v}}$

$r \updownarrow X$, $r_0 \updownarrow P_0$, $t \updownarrow \lambda$



$X = (x, y, z)$ any point on the $\overline{OX}$-axis

Let's remember the equation of the axis.

$X = (x, y, z)$ is a point of $\overline{OX}$ if

$\boxed{\overrightarrow{OX} = \lambda\vec{i}}$ $\lambda \in \mathbb{R}$

$X - O = \lambda\vec{i}$

$X = O + \lambda\vec{i} \quad \therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

So the coordinates of the generic point X that belongs to the $\overline{OX}$-axis must satisfy $x = \lambda \in \mathbb{R}$ (

$\circledast \begin{cases} x = 0 + \lambda \\ y = 0 + 0 \\ z = 0 + 0 \end{cases}$

belongs to the $\overline{OX}$-axis must satisfy

$$\left.\begin{array}{l} x = \lambda \in \mathbb{R} \\ y = 0 \\ z = 0 \end{array}\right\} \Rightarrow P = (x, 0, 0), \text{ with } \underline{x = \lambda \in \mathbb{R}}.$$

(this is what we have done at the beginning of $\underline{\S 10.1}$.

2) When we open the vector parametric equations of the straight line we obtain another possible formulation of its equation: as we did in ⊛

$\overrightarrow{P_0P} = \lambda\vec{v}$ $\quad P \equiv$ generic point of r $\quad P = (x, y, z)$

$P - P_0 = \lambda\vec{v}$ $\quad P_0 =$ a given known point of r $\quad P_0 = (x_0, y_0, z_0)$

$P = P_0 + \lambda\vec{v}$ $\quad v =$ direction vector of r $\quad \vec{v} = (a, b, c)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_0 + \lambda a \\ y_0 + \lambda b \\ z_0 + \lambda c \end{pmatrix} \Rightarrow ⊛⊛ \begin{cases} x = x_0 + \lambda a \\ y = y_0 + \lambda b \\ z = z_0 + \lambda c \end{cases}$$

$\lambda \in \mathbb{R}$: free parameter

⊛⊛ This is called the Scalar parametric equations of r.

3) And the third possible formulation is when we isolate the parameter λ in all the scalar parametric equations

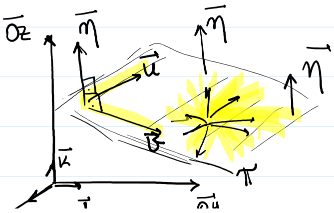
$$\begin{cases} x = x_0 + \lambda a & \Rightarrow \quad \lambda = \dfrac{x - x_0}{a} \\ y = y_0 + \lambda b & \Rightarrow \quad \lambda = \dfrac{y - y_0}{b} \\ z = z_0 + \lambda c & \Rightarrow \quad \lambda = \dfrac{z - z_0}{c} \end{cases}$$

Thus

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}} \quad \boxed{(\text{if } a \neq 0,\ b \neq 0\ \&\ c \neq 0)}$$

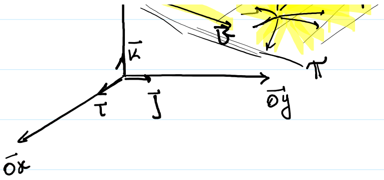
is called the Standard form for the equations of the straight line through (x_0, y_0, z_0), parallel to $\vec{v} = (a, b, c)$

Planes in $\mathbb{R}^3$

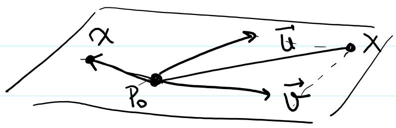


$\vec{n}$ = normal vector of the plane π.

This means that



This means that any linear combination of $\vec{u}$ and $\vec{v}$ is still perpendicular to $\vec{\eta}$.



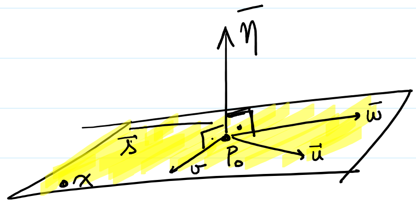
1) $\vec{v}$ & $\vec{u}$ all called the direction vectors of the plane Π.

2) $P=(x_0,y_0,z_0)$ is a known point of Π.

3) $X=(x,y,z)$ is a general point of Π if $\boxed{\overrightarrow{P_0X} = \lambda\vec{u} + \beta\vec{v} \quad \text{for } \lambda\in\mathbb{R},\ \beta\in\mathbb{R}}$

This is the vector parametric equation of Π.

Π.



If we now consider the fact that every vector on Π is perpendicular to $\vec{\eta}$, we can obtain another formulation of the equation of Π.

But one observation The Equation of Π is satisfied by points of Π.

So: Consider $P_0=(x_0,y_0,z_0)$ (known point of Π)
$P=(x,y,z)$ (generic point of Π)

$$\overrightarrow{P_0P} = P - P_0 = (x,y,z)-(x_0,y_0,z_0) = (x-x_0,\ y-y_0,\ z-z_0)$$

This is a vector on Π.

This is general vector of Π, which is perpendicular to the vector $\vec{\eta} \equiv$ normal to Π.

Recall: $\boxed{\vec{u}\perp\vec{v} \iff \vec{u}\cdot\vec{v}=0}$

So $\overrightarrow{P_0P}\perp\vec{\eta} \iff \overrightarrow{P_0P}\cdot\vec{\eta}=0 \iff$

$$(x-x_0,\ y-y_0,\ z-z_0)\cdot(a,b,c) = 0 \iff$$

Standard Equation

$$ax+by+cz-(ax_0+by_0+cz_0)=0 \iff$$

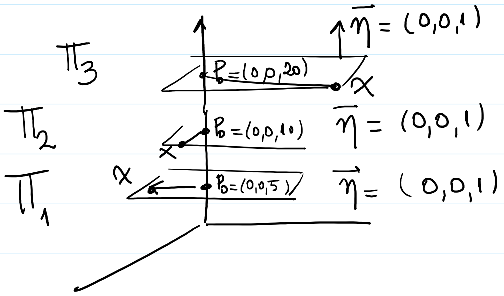
Standard Equation of the plane Π

$$ax+by+cz-(ax_0+by_0+cz_0)=0 \iff$$

$$\boxed{ax+by+cz = d}$$

$(d = (ax_0+by_0+cz_0))$

d is responsible for localizing the plane Π, since all parallel planes are orthogonal to the same $\vec{\eta}$ = normal vector.



$\begin{cases}\alpha=5\\ \alpha=10\\ \alpha=15\end{cases}$

$$\vec{P_0X} = X-P_0 = (x,y,z)-(0,0,\alpha) = (x-0,\, y-0,\, z-\alpha)$$

$\vec{\eta} = (0,0,1)$ normal vector to Π_1, Π_2, Π_3

So $\vec{P_0X}\cdot\vec{\eta} = (x,y,z-\alpha)\cdot(0,0,1) = 0$

$$0+0+z-\alpha = 0$$

$$\boxed{z=\alpha} \quad \begin{cases} z=5\\ z=10\\ z=20\end{cases}$$

These are the standar equations of each one of the planes Π_1, Π_2 & Π_3

all are parallel to the coordinate plane $\boxed{z=0}$ which has the origin $P_0=(0,0,0)$ as known point.