In this extra material, I would like to finally answer the question about the types of equations of straight lines when we are in $\Pi = \mathbb{R}^2 = \Pi_{xy}$ or in $\mathbb{R}^3$.

So: Given a known Point $P_0 = (x_0, y_0, z_0)$ & a direction $\vec{v} = (a, b, c)$ we can construct the equations of the straight line r assuming that:

1) Any Point X that belongs to the line r will form a oriented segment $\overline{P_0X}$ parallel to $\vec{v}$.
So

$$\boxed{\overrightarrow{P_0X} = \lambda \vec{v}} \iff \boxed{X - P_0 = \lambda \vec{v}} \iff \boxed{X = P_0 + \lambda \vec{v}}$$

vector parametric eq of r.

2) $$X = P_0 + \lambda \vec{v} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \iff$$

$$\begin{cases} x = x_0 + \lambda a \\ y = y_0 + \lambda b \\ z = z_0 + \lambda c \end{cases}$$

$\lambda \in \mathbb{R}$ $\lambda \in (-\infty, +\infty)$

These are the Scalar parametric equations of r.

3) And isolating λ on (2) we have:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

the standard form of the equations of r.

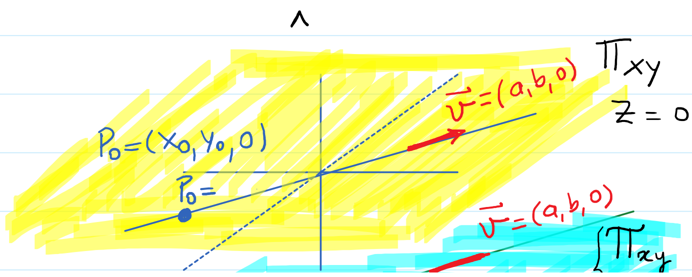
$(a \neq 0, b \neq 0, c \neq 0)$

But now, let's think in the case when $\boxed{c = 0}$

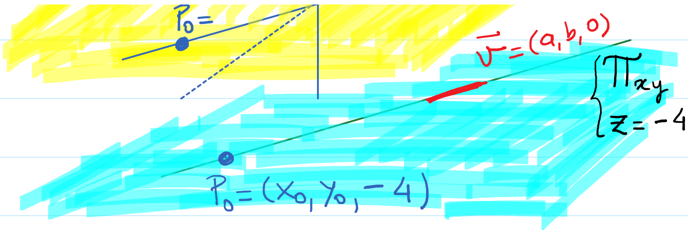
So. the equation of r is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

$$\begin{cases} x = x_0 + \lambda a \\ y = y_0 + \lambda b \\ z = z_0 \end{cases}$$



$$\begin{cases} y = y_0 + \lambda b \\ z = z_0 \end{cases}$$



And Now, we have to modify the standard form of the line equation since $\underline{c = 0}$

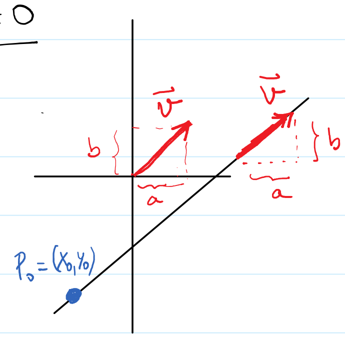
$$\begin{cases} \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \cancel{\dfrac{z - z_0}{c}} \\ z = z_0 \end{cases}$$

Π_{xy} : $\begin{cases} \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} \\ z = 0 \end{cases}$ This is a line on the plane we associated to $\mathbb{R}^2$ originally.

$\tilde{\Pi}_{xy}$ $\begin{cases} \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} \\ z = -4 \end{cases}$ This is another line now on $\tilde{\Pi}_{xy}$. $\tilde{\Pi}_{xy} \parallel \Pi_{xy}$. $\tilde{\Pi}_{xy}$ is passing through the point $P = (0, 0, -4)$

__Now__ let's look closer to this standard form of the equation of r in $\mathbb{R}^2 = \Pi_{xy}$, $z = 0$.

$\underline{z = 0}$



$$\frac{\Delta y}{\Delta x} = \frac{b}{a} = \alpha$$

If we write the eq of line as we are use to, we have:

$$y - y_0 = \alpha (x - x_0)$$

$$y - y_0 = \frac{b}{a} (x - x_0)$$

$$a(y - y_0) = b(x - x_0)$$

$$\Longrightarrow \quad \frac{(y - y_0)}{b} = \frac{(x - x_0)}{a}$$

$$\frac{x - x_0}{a} = \lambda \qquad \Longrightarrow$$

$$\frac{y - y_0}{b} = \lambda$$

$$x - x_0 = \lambda a$$

$$y - y_0 = \lambda b$$

$$\begin{cases} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \end{pmatrix} \\ z = 0 \end{cases}$$

$$\overline{b} - \overline{a}$$

$$\therefore \quad \frac{(x-x_0)}{a} = \frac{(y-y_0)}{b}, \qquad \begin{cases} (y) - (y_0) \ldots (b) \\ z = 0 \\ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \end{cases}$$

with $z=0$ or $z=-4$ or $z=k$ of any height we choose to form a plane parallel to Π_{xy} but passing through $P_0 = (x_0, y_0, k)$

So $\quad \dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} \Rightarrow \dfrac{b}{a}(x-x_0) = y - y_0 \Rightarrow$

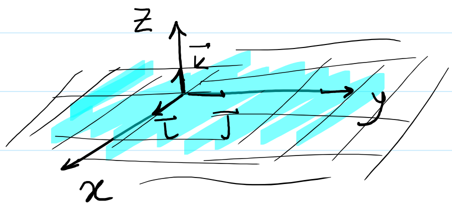
$$\boxed{y = y_0 + \left(\frac{b}{a}\right)(x - x_0)}$$

slope of our $\vec{v} \equiv$ direction vector of r.

- Now let's think again about the equation of the axis

$\overrightarrow{Oy}$
$x = 0$
y free

$\begin{cases} x \text{ free} \\ y = 0 \end{cases}$ is the equation of $\overrightarrow{Ox}$



But we have to remember that $z = 0$, since we are in $\Pi_{xy} = \mathbb{R}^2$.

To obtain the equation of $\overrightarrow{Ox}$,

we want $P_0 = (0,0,0)$ point of our line r.
$\vec{i} = (1,0,0)$ direction vector of our line.

$$\overrightarrow{XP_0} = \lambda \vec{i} \Rightarrow X - P_0 = \lambda \vec{i} \Rightarrow X = P_0 + \lambda i$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 + \lambda \\ 0 + 0 \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$$

$x = \lambda \leftarrow x$ is free to assume any value in $\mathbb{R}$
$\boxed{y = 0}$ fixed in $\mathbb{R}^2$
$\boxed{z = 0}$ fixed because we are seeing $\mathbb{R}^2$ inside $\mathbb{R}^3$

$\boxed{y = 0}$ fixed in $\mathbb{R}^2$

$\boxed{z = 0}$ fixed because we are seeing $\mathbb{R}^2$ inside $\mathbb{R}^3$.

Analogously, we have

$\overline{Oy}$: $P = (0,0,0)$ $\vec{v} = \vec{j} = (0,1,0)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \cdot 1 \\ 0 \end{pmatrix}$$

So $x = 0$, $y =$ free, $z = 0$ is the equation of $\overline{Oy}$.

& the standard form in $\mathbb{R}^3$ is

$$x = 0 \quad z = 0 \quad \boxed{\frac{y - 0}{1} = \lambda} \quad (y - 0 = 1 \cdot \lambda) \quad (y = \lambda)$$

I hope now I have answered the question in a more detailed way. ☺

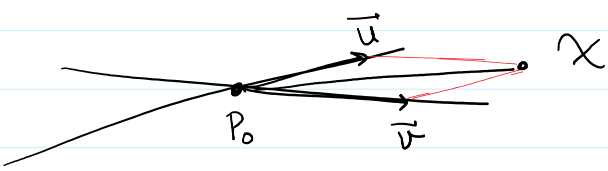
II. About π and its vectorial equation.

* Obs 1. Two vectors not parallel define a plane π

Obs 2 To be able to localize this plane in space we need to know a point $P_0 = (x_0, y_0, z_0)$ that belongs to π.

Otherwise, we have a family of infinite planes (all parallel to each other)

π: $P_0 = (x_0, y_0, z_0)$ $\bar{u} = (a, b, c)$ $\bar{v} = (m, n, p)$



Every point $x \in \pi$, forms a vector $\overrightarrow{P_0 x}$

which is a linear combination of $\vec{u}$ & $\vec{v}$.

$$\overrightarrow{P_0X} = \alpha\vec{u} + \beta\vec{v}$$

$$X - P_0 = \alpha\vec{u} + \beta\vec{v}$$

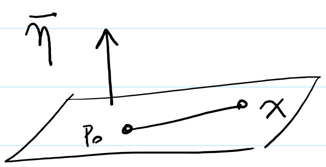
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \alpha\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\text{straight line } r} + \underbrace{\beta\begin{pmatrix} m \\ n \\ p \end{pmatrix}}_{\text{straight line } s}$$

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = r \cap s = \text{intersection of two lines in } \underline{\mathbb{R}^3}$$

So, this is another way to think about planes in $\mathbb{R}^3$.

III π & the point-normal equation.

$\left\{ \begin{array}{l} P_0 \text{ given} \\ \vec{\eta} \text{ given} \end{array} \right.$



$$\overrightarrow{P_0X} \perp \overrightarrow{\eta}$$

$$(x - x_0, y - y_0, z - z_0) \perp (a, b, c) \implies$$

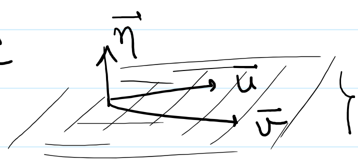
$$(x - x_0, y - y_0, z - z_0) \bullet (a, b, c) = 0 \implies$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = (ax_0 + by_0 + cz_0) = d$$

- Here (x, y, z) are the coordinates of the generic point X which belongs to π.

- $\vec{\eta} = (a, b, c) = \vec{u} \times \vec{v}$



$\}$ any two vectors of π, that are NOT parallel.

Obs. $2x + 4y + 7z = -1 \quad \Pi \subseteq \mathbb{R}^3$

To find points that belong to Π,

we choose 2 variables, and obtain the third satisfying the equation:

For example:

$P_1 = (0, 0, z)$ z being such that

$$2(0) + 4(0) + 7z = -1 \Rightarrow z = -1/7$$

$P_1 = (0, 0, -1/7)$

$P_2 = (x, 0, 0)$ x being such that

$$2x + 4(0) + 7(0) = -1 \Rightarrow x = -1/2$$

So $P_2 = (-1/2, 0, 0)$

Och $P_3 = (1, y, 1)$, y being such that

$$2(1) + 4y + 7(1) = -1$$

$P_3 = (1, -\frac{5}{2}, 1)$

$$4y = -1 - 2 - 7 \Rightarrow y = -\frac{10}{4}$$

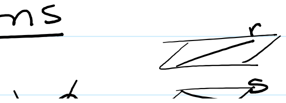
$$y = -\frac{5}{2}$$

So P_1, P_2 & P_3 are 3 point in $\mathbb{R}^3$.

so $\overrightarrow{P_1P_2}$ & $\overrightarrow{P_1P_3}$ for example can generate 2 vectors (not parallel) in Π.

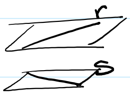
So $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \lambda \vec{n} := \lambda(2, 4, 7)$

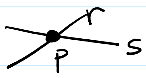
Intersections



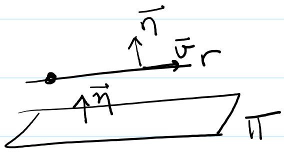
Intersections

1) $r \cap s = \begin{cases} \emptyset \\ P \end{cases}$

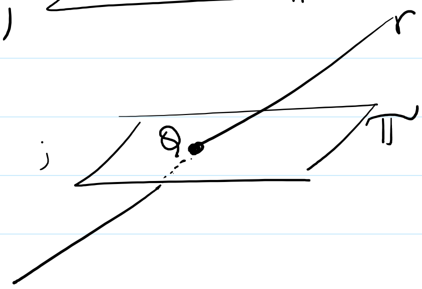




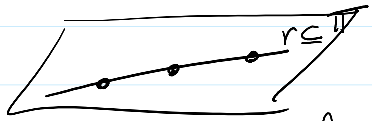
2) $r \cap \Pi = \begin{cases} \emptyset, \\ P \ ; \\ r \ ; \text{ if } r \subseteq \Pi \end{cases}$



$\therefore$ system without solution



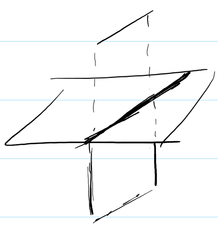
system with unique solution

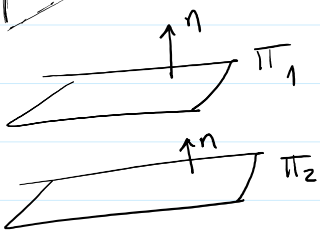


system with infinit solution.

This mean there is one free parameter to express the solution.

3) $\Pi_1 \cap \Pi_2 = \begin{cases} r = \\ \emptyset \\ \Pi_1 = \Pi_2 \end{cases}$





Parallel Π_1 & Π_2

the equation are <u>multiple</u>

Example 6 page 597

Find the intersection between $\Pi_1 : x + y - z = 0$

& $\Pi_2 : y + 2z = 6$.

& $\Pi_2: \; y + 2z = 6.$

$$\begin{aligned} \text{Eq}(1) &\left\{ \begin{array}{l} x + y - z = 0 \\ 0x + 1y + 2z = 6 \quad \Rightarrow \quad y = 6 - 2z \end{array} \right. \\ \text{Eq}(2) & \end{aligned}$$

I will substitute this in eq (1)

$$x + (6 - 2z) - z = 0$$

$$x + 6 - 3z = 0 \quad \Rightarrow \quad x = -6 + 3z$$

Now we observe that both x & y depend on z.

$$x = -6 + 3z$$

$$y = +6 - 2z$$

$$z = 0 + 1z \qquad \boxed{z = z} \boxed{= \lambda} \text{ because !}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ 0 \end{pmatrix} + \begin{pmatrix} 3\lambda \\ -2\lambda \\ 1\lambda \end{pmatrix}$$

$$\underset{\substack{\uparrow \\ \text{generic} \\ \text{point of } r}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underbrace{\begin{pmatrix} -6 \\ 6 \\ 0 \end{pmatrix}}_{\text{point of } r} + \lambda \underbrace{\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}}_{\text{direction vector of } r}$$

$$\overbrace{\frac{x+6}{3}}^{=\lambda} = \overbrace{\frac{y-6}{-2}}^{=\lambda} = \overbrace{\frac{z-0}{1}}^{=\lambda} \quad \Downarrow$$

$$\boxed{\frac{x+6}{3} = \frac{6-y}{2} = z}$$

$$r = \Pi_1 \cap \Pi_2$$