Lines_Distances_Intersection

den 8 oktober 2020 18:11

I hope to help with the two next exercises:

1) Given 2 lines on Π_{xy} $(z=0)$, find their intersection.

$$l_1: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad l_2: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

<u>I. Resolution option</u>

l_1 & l_2 are lines on $\mathbb{R}^2$, so we could find their standard forms and obtain the intersection as a solution of a linear system.

l_1. $\quad \begin{aligned} x &= 1 + 3\alpha \\ y &= 0 + 1\alpha \end{aligned} \quad \Rightarrow \quad \frac{x-1}{3} = \alpha \quad \& \quad \frac{y-0}{1} = \alpha$

The standard form of the equation is $\dfrac{x-1}{3} = \dfrac{y-0}{1}$

Simplifying $\Rightarrow$ $\quad y = \frac{1}{3}x - \frac{1}{3} \quad$ or $\quad \boxed{x - 3y = -1}$

l_2: $\quad \begin{aligned} x &= 0 + 1\lambda \\ y &= -1 + 3\lambda \end{aligned} \quad \Rightarrow \quad \frac{x-0}{1} = \lambda \quad \& \quad \frac{y+1}{3} = \lambda$

The standard form of the line equation is

$$\frac{x-0}{1} = \frac{y+1}{3} \Rightarrow 3(x-0) = y+1 \Rightarrow y = 3(x-0) - 1$$

or $\boxed{3x - y = 1}$

Solving the linear system, we have:

$$(1)\begin{cases} x - 3y = 1 \\ 3x - y = 1 \end{cases} \qquad (-3)\begin{cases} x - 3y = 1 \\ 3x - y = 1 \end{cases}$$

$$\diagup \quad 8y = -2 \qquad y = -\frac{1}{4}$$

Substituting $y = -\frac{1}{4}$ in eq (1), we have

$x - 3\left(-\frac{1}{4}\right) = 1 \Rightarrow x = 1 - \frac{3}{4} = \frac{1}{4}$ $\qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix}$ but remember $z = 0$

So in $\mathbb{R}^3$ the answer is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/4 \\ 0 \end{pmatrix}$

II. Another possibility to solve:

$$l_1: \begin{cases} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}\alpha \\ z = 0 \end{cases} \qquad l_2: \begin{cases} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ z = 0 \end{cases}$$

$$\underbrace{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + 3\alpha \\ 0 + \alpha \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 + \lambda \\ -1 + 3\lambda \end{pmatrix}}$$

Comparing both formulations: for the generic point $\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 1 + 3\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} \lambda \\ -1 + 3\lambda \end{pmatrix} \Rightarrow \begin{cases} 1 + 3\alpha = \lambda \\ \alpha = -1 + 3\lambda \end{cases}$$

substituting

$$1 + 3(-1 + 3\lambda) = \lambda$$

$$1 - 3 + 9\lambda = \lambda \Rightarrow -2 = \lambda - 9\lambda = -8\lambda \Rightarrow \lambda = \frac{1}{4}$$

substituting now in the previous equation

$$\alpha = -1 + 3 \cdot \left(\frac{1}{4}\right) = \frac{-4 + 3}{4} = -\frac{1}{4}$$

So with α we generate the intersection point via l_1 and with λ we generate the intersection point via l_2.

$$l_1: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + 3(-1/4) \\ 0 + (-1/4) \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix}$$

$$l_2: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 + 1/4 \\ -1 + 3(1/4) \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix}$$

the same point P belongs to l_1 & l_2. So $P \in l_1 \cap l_2$.

Remember the equations are in $\mathbb{R}^3$. and $z = 0$

so $P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/4 \\ 0 \end{pmatrix}$

$$r = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1/4\\-1/4\\0\end{pmatrix}$$

Ex 2. Find the distance between 2 lines

$$l_1: \begin{cases} x - 3y = 1 \\ z = 4 \end{cases} \qquad l_2: \begin{cases} -3x + 9y = 2 \\ z = 4 \end{cases}$$

1^{st} Resolution

l_1: the equation is almost in the standard form.

So, initially we put the equation in the correct standard form.

$$\begin{cases} x - 1 = 3y \\ z = 4 \end{cases} \Rightarrow \frac{x-1}{3} = \frac{y-0}{1} \; ; \; z = 4 \Rightarrow \begin{aligned} x - 1 &= 3\lambda \\ y - 0 &= 1\lambda \\ z &= 4 \end{aligned}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\4\end{pmatrix} + \lambda \underbrace{\begin{pmatrix}3\\1\\0\end{pmatrix}}_{\vec{v}}$$

l_2: The same way for l_2.

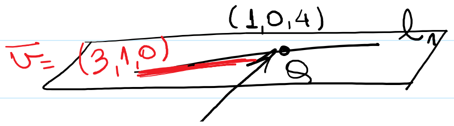
$$\begin{cases} -3x + 9y = 2 \\ z = 4 \end{cases} \qquad 9y = 3x + 2 = 3\left(x + 2/3\right)$$

$$\frac{\cancel{3}\left(x - (-2/3)\right)}{9} = \frac{\cancel{9}y}{3} \Rightarrow \begin{aligned} \frac{x - (-2/3)}{9} &= \lambda \\ \frac{y - 0}{3} &= \lambda \\ z &= 4 \end{aligned} \Rightarrow \begin{aligned} x - (-2/3) &= 9\lambda \\ y - 0 &= 3\lambda \\ z &= 4 \end{aligned}$$

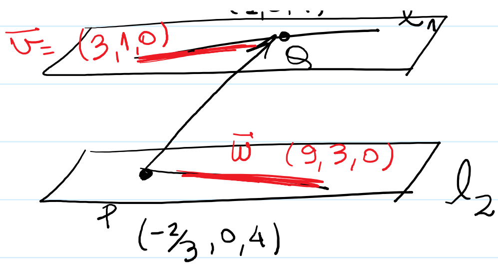
$z = 4$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2/3\\0\\4\end{pmatrix} + \lambda \underbrace{\begin{pmatrix}9\\3\\0\end{pmatrix}}_{\vec{w}}$$

Now we can compute the distance



Obs: $(3,1,0) \parallel (9,3,0)$



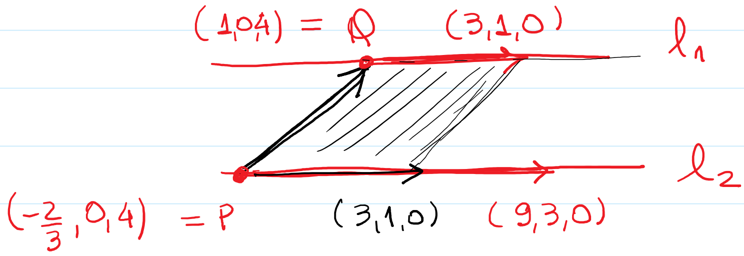
Obs: $(3,1,0) \,//\, (9,3,0)$

$(9,3,0) = 3(3,1,0)$

so $\vec{v} \,//\, \vec{w}$

$\vec{v} \times \vec{w} = \vec{0}$

In fact, both lines are parallel and are in the same plane:



$\text{Area} = \|\vec{PQ} \times \vec{v}\|$

$\text{Base} = \|\vec{v}\|$

$$d(\ell_1, \ell_2) = \frac{\text{Area}}{\text{Base}} = \frac{\sqrt{(5/3)^2 + 0^2 + 0^2}}{\sqrt{9+1+0}} = \frac{5/3}{\sqrt{10}} = \frac{5}{3\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\cancel{5}\sqrt{10}}{3\cancel{0}} = \frac{\sqrt{10}}{6}$$

$$\vec{PQ} = \left(1 - \left(-\frac{2}{3}\right),\ 0-0,\ 4-0\right) = \left(5/3, 0, 0\right)$$

$$\vec{PQ} \times (3,1,0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5/3 & 0 & 0 \\ 3 & 1 & 0 \end{vmatrix} = \vec{i}\,(0) - \vec{j}\,(0) + \vec{k}\left(\frac{5}{3}\right)$$

$$\vec{PQ} \times (3,1,0) = (0, 0, 5/3)$$

$$\boxed{d(\ell_1, \ell_2) = \frac{\sqrt{10}}{6}}$$