F. 7.1 Book Lay Chapter 1. Linear Systems

§1.1 & §1.2

pag 20

→ Solution by Elementary Row operations.

Examples:

1) Solve the systems and give an interpretation for the solution.

1.a) $\begin{cases} x - 3y = 1 \\ 3x - y = 1 \end{cases}$ 1.b) $\begin{cases} x - 3y = 1 \\ -3x + 9y = 2 \end{cases}$ $\left(-\frac{1}{3}\right)$ 1.c) $\begin{cases} x - 3y = 1 \\ -3x + 9y = -3 \end{cases}$ $\left(-\frac{1}{3}\right)$

1.a)

$$\begin{cases} -3x - 3(-3y) = -3(1) \\ 3x - y = 1 \end{cases}$$

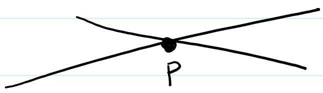
$$8y = -2$$

$$\boxed{y = -\frac{1}{4}}$$

$3x - (-1/4) = 1$

$3x = 1 - 1/4 = 3/4 \Rightarrow x = 1/4$

$P = (x, y) = (1/4, -1/4) = l_1 \cap l_2$



Unique solution of the system

1.b)

$$\begin{cases} x - 3y = 1 \\ x - 3y = -2/3 \end{cases}$$

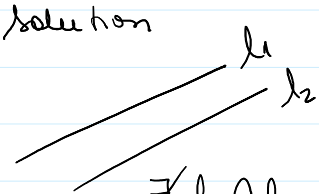
$x = 3y - 2/3$

$(\cancel{3y} - 2/3) - \cancel{3y} = 1$

$-\frac{2}{3} = 1$

???

The system has NO solution

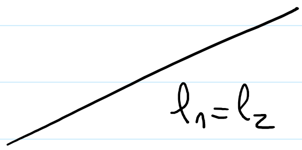


$\nexists\, l_1 \cap l_2$

1.c)

$$\begin{cases} x - 3y = 1 & l_1 \\ x - 3y = 1 & l_2 \end{cases}$$

$l_1 = l_2$ $l_1 \cap l_2 = l_1 = l_2$



The system has infinite solutions ∴ all points of l_1 satisfy the system.

A linear system can be written in Matrix Notation:

$$\begin{cases} x - 3y = 1 \\ 3x - y = 1 \end{cases} \Leftrightarrow \underbrace{\begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} \Leftrightarrow AX = B$$

$$\underset{2\times 2}{A} \cdot \underset{2\times 1}{X} = \underset{2\times 1}{B}$$

$A_{2\times 2}$ matrix of coeficients

$B_{2\times 1}$ = vector with the independent term

$X_{2\times 1}$ = solution vector to be obtained.

General Case $A \cdot X = B$

$X_{2\times 1}$ = solution vector to be ...

General Case
$$A_{m\times n} \cdot X_{n\times 1} = B_{m\times 1}$$

Exemple 2

$\exists \cap ?$

$$\begin{cases} \pi_1: & 1x - 2y + 1z = 0 \\ \pi_2: & 0x + 2y - 8z = 8 \\ \pi_3: & 5x - 5z = 10 \end{cases}$$

$\Updownarrow$

Matrix Notation: $Ax = B$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}$$

$\Updownarrow$ Augmented Matrix: $[A \mid B]$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

Types of solution:

(1) $\pi_1 \cap \pi_2 \cap \pi_3 = P =$ point

unique solution

(2) $\pi_1 \cap \pi_2 \cap \pi_3 =$ line or plane

Infinite solution

(3) $\pi_1 \parallel \pi_2 \parallel \pi_3 \wedge \pi_i \cap \pi_j = \emptyset \quad \forall i \neq j$

There is no solution

Solving a Linear System with Elementary Row Operations:

(1) Replacement: Replace 1 Row by the sum of itself and a multiple of another row.

(2) Interchange: Interchange 2 Rows.

(3) Multiply a Row by a nonzero constant.

Result: $Ax = B \wedge \tilde{A}x = \tilde{B}$ both linear systems

theorem

If $[\tilde{A} \mid \tilde{B}]$ the augmented matriz is obtained by Elementary Row Operations from $[A \mid B]$

Then both systems have the same solution X.

Then both systems have the same solution x.

Methodology (1) (Gaussian Elimination)

1) Start from the augmented matrix $[A|B]$
2) Apply Elementary Row Operations to $[A|B]$ to obtain $[\tilde{A}|\tilde{B}]$ $\tilde{A}$ = triangular matrix
3) Apply Backward substitution on $\tilde{A}x=\tilde{B}$ to obtain the solution:

Back to our 3-D example.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] \begin{array}{l} \leftarrow L_1 = \text{Pivot line} \quad A_{11}=1\neq 0 \ \text{Pivot (important)} \\ \leftarrow L_2 \leftarrow \frac{1}{2}L_2 \\ \leftarrow L_3 \leftarrow L_3 - 5L_1 \end{array}$$

Ready

A, B

Nothing to do:

The operations use the Pivot line

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & \\ 0 & 1 & -4 & \\ 5-5(1) & 0-5(-2) & -5-5(1) & 10-5(0) \end{array}\right] \quad (L_3 \leftarrow L_3 - 5L_1)$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{array}\right]$$

New Pivot Line: Goal transform to zero all values below the pivot in the same column.

$L_3 \leftarrow \frac{1}{10}L_3$ (just to simplify)

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$L_2 \equiv$ Pivot line (fixed)

$L_3 \leftarrow L_3 - L_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0-0 & 1-1 & -1-(-4) & 1-(4) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 3 & -3 \end{array}\right] \quad L_3 \leftarrow \tfrac{1}{3}L_3$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 3 & -3 \end{array}\right] \quad L_3 \leftarrow \frac{1}{3}L_3$$

Upper Triangular form

$$\underbrace{\left[\begin{array}{ccc}1 & -2 & 1\\ 0 & 1 & -4\\ 0 & 0 & 1\end{array}\right.}_{\tilde{A}}\underbrace{\left|\begin{array}{c}0\\4\\-1\end{array}\right]}_{\tilde{B}} \qquad [\tilde{A} \mid \tilde{B}] \quad \tilde{A} = \text{Upper Triangular form}$$

3) Now we come back to the matrix notation $\tilde{A}X = \tilde{B}$ and compute the solution by backward substitution.

$$\overbrace{\begin{pmatrix}1 & -2 & 1\\ 0 & 1 & -4\\ 0 & 0 & 1\end{pmatrix}}^{\tilde{A}}\begin{pmatrix}x\\y\\z\end{pmatrix} = \overbrace{\begin{pmatrix}0\\4\\-1\end{pmatrix}}^{\tilde{B}} \qquad \begin{cases} x - 2y + z = 0 \\ \quad y - 4z = 4 \\ 0 \quad 0 \quad \boxed{z = -1} \end{cases}$$

substitute.

$y = 4 + 4z$

$y = 4 + 4(-1) = 0$

$\boxed{y = 0}$

$\Rightarrow \; x = 2y - z + 0$

$x = 2(0) - 1(-1) + 0$

$x = 0 + 1 + 0 = 1$

$\boxed{x = 1}$

So, the system solution is $\boxed{P = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}}$

unique solution.

Methodology (2)

Obs: Here we are assuming that $\tilde{A}$ can be equal I.

1) Start with the $[A \mid B]$ augmented matrix

2) Apply the Elementary Row Operations to obtain $[\tilde{A} \mid \tilde{B}]$ such that $\tilde{A} = [I] =$ identity (*) (OR ECHELON FORM)

3) If the solution exists, It will be obtained from $\tilde{B}$.

Exemple: Revisiting the example we solved before:

1) $$\left[\begin{array}{ccc|c}1 & -2 & 1 & 0\\ 0 & 2 & -8 & 8\\ 5 & 0 & -5 & 10\end{array}\right]$$

* $$\left[\begin{array}{ccc|c}1 & -2 & 1 & 0\\ 0 & 1 & -4 & 4\end{array}\right. \quad \begin{array}{l} L_1 \leftarrow L_1 - L_3 \\ L_2 \leftarrow L_2 + 4L_3 \end{array}$$ goal

Transform to zero the values above the pivot on the pivot column.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & \\ 0 & 1 & -4 & 4 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right] \quad \begin{array}{l} L_2 \leftarrow L_2 + 4L_3 \\ L_3 \leftarrow \text{Pivot line} ;\ a_{3\times3} = 1 \neq 0 \end{array}$$

the values ... on the pivot column.

$$\left[\begin{array}{ccc|c} 1-0 & -2-0 & 1-1 & 0-(-1) \\ 0 & 1 & -4+4(1) & 4+4(-1) \\ 0 & 0 & 1 & -1 \end{array}\right] =$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} L_1 \leftarrow L_1 + 2L_2 \\ \leftarrow L_2 : \text{New Pivot} \end{array}$$

Unique solution!

$$\underbrace{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right.}_{I} \underbrace{\left|\begin{array}{c} 1 \\ 0 \\ -1 \end{array}\right]}_{\text{Solution}} \Rightarrow \begin{cases} 1x = 1 \\ 1y = 0 \\ 1z = -1 \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Lay Page 23

Two fundamental Questions about a linear system:

1. Is the system CONSISTENT?
   Does it have at least one solution?

2. If the solution exists,
   is it one solution?
   or there will be infinite solutions?

The Natural answer is obtained after we obtain the ROW ECHELON FORM of the augmente matrix of the system.

It is the best. Reduced form of a augmented matrix obtained by low elementary operations.

* Noll rows efter elementary operations MUST be placed bellow Nonzero rows.

* The first element of a non-zero row is allowed to be zero, If all elements bellow it in its column ...

Page 30 - Example 2 Lay.

$Ax = B \rightarrow [A \mid B]$

?

$$\left[\begin{array}{cccc|c} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right]$$

to be zero, if all elements bellow it in its column are zero.

To start the process, the pivot MUST BE $\neq 0$.

$L_1 \leftrightarrow L_4$ exchange Rows.

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right] \quad \begin{array}{l} L_1 \leftarrow \text{Pivot} \\ L_2 \leftarrow L_2 + 1L_1 \\ L_3 \leftarrow L_3 + 2L_1 \\ L_4 \text{ ready} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right] \quad \begin{array}{l} L_2 \leftarrow \frac{1}{2} L_2 \\ L_3 \leftarrow \frac{1}{5} L_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right] \quad \begin{array}{l} L_2 \leftarrow \text{Pivot} \\ L_3 \leftarrow L_3 - L_2 \\ L_4 \leftarrow L_4 + 3L_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{array}\right] \quad L_4 \leftarrow -\frac{1}{5} L_4 \qquad L_3 \leftrightarrow L_4$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$B_1 \quad B_2$

Obs we can compute the solution from this $\tilde{A}|\tilde{B}$

OR we can obtain the Reduced ECHELON FORM.

Assuming now the pivot from the bottom part & performing the Row operations to create null values above the pivot on its Column.

Theorem 2 (Lay)

1) $Ax = B$ is consistent $\Leftrightarrow$ The ECHELON FORM of the augmented matrix Has NO Row $[0\ 0\ .\ 0 \mid b]$ with $b \neq 0$

2) $Ax = B$ is consistent $\Rightarrow$ $Ax = b$ has unique solution when there is no free parameter on the solution $\therefore$ $|\det(A)| \neq 0$.

OR

$Ax = b$ has infinite solutions. with at least one variable being a free parameter.
$\det A = 0$
$[0 \ldots 0 \mid 0]$ in the Echelon form.

Page 34: Solution of linear systems

$$\begin{array}{c} \quad\overbrace{\qquad\qquad\qquad}^{\tilde{A}} \quad \overbrace{\quad}^{\tilde{B}} \\ \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \quad x \quad\ y \quad\ z \end{array}$$

$\tilde{A}x = \tilde{B}$

$$\begin{cases} x + 0y - 5z = 1 \\ 0 + 1y + z = 4 \\ 0x + 0y + 0z = 0 \end{cases}$$

Backward substitution

$\boxed{z \text{ is free}}$

Solution: $\boxed{y = 4 - z}$ free $z = \lambda$

$x = 1 + 5z$ free $z = \lambda$

$\boxed{z = \lambda}$, $\lambda \in \mathbb{R}$.

$x = 1 + 5\lambda$
$y = 4 - 1\lambda$
$z = 0 + 1\lambda$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 5\lambda \\ -1\lambda \\ \lambda \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$$

eq of a line in $\mathbb{R}^3$

Solutions

$\lambda = 0 \Rightarrow P_1 = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$

$\lambda = 1 \Rightarrow P_2 = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$

$\lambda = -1$

$P_3 = \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}$

eq of a line in $\mathbb{R}^3$

Ex 2

Ex 4 page 34 Lay

$$\left[\begin{array}{ccccc|c} \boxed{1} & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & \boxed{1} & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & \boxed{1} & 7 \end{array}\right]$$

$$\begin{cases} 1x_1 + 6x_2 + 0x_3 + 3x_4 + 0x_5 = 0 \\ 1x_3 - 4x_4 = 5 \\ 1x_5 = 7 \end{cases}$$

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix}$$

Pivot columns are 1, 3, 5 so variables x_1, x_3, x_5 are defined. x_2 & x_4 are free

free parameters.

$$\begin{cases} x_5 = 7 \\ 1x_3 = 5 + 4x_4 \\ 1x_1 = 0 - 3x_4 - 6x_2 \end{cases}$$

$x_2 = \lambda$

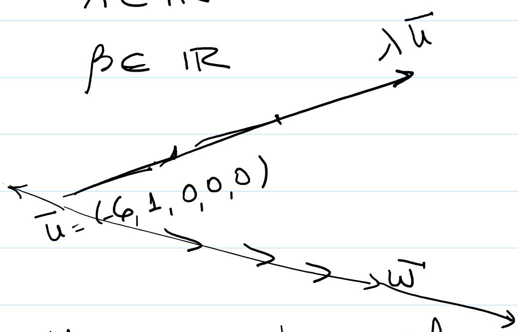
$x_4 = \beta$

$$\begin{aligned} x_1 &= 0 - 6\lambda - 3\beta \\ x_2 &= 0 + 1\lambda + 0\beta \\ x_3 &= 5 + 0\lambda + 4\beta \\ x_4 &= 0 + 0\lambda + 1\beta \\ x_5 &= 7 + 0\lambda + 0\beta \end{aligned}$$

$\lambda \in \mathbb{R}$

$\beta \in \mathbb{R}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{pmatrix} + \lambda \overbrace{\begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}^{\bar{u}} + \beta \overbrace{\begin{pmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix}}^{\bar{w}}$$



part of the solution $\lambda = 0, \beta = 0$

They are the generators of the solution