Mål: §10.4 Adams Bok: Typiska problem i Analytisk Geometri

1. • Obtain the Echelon form of the augmented matrix of the system:
   • Find its solution
   • give a geometric interpretation

$$\begin{matrix} \Pi_1: \\ \Pi_2: \\ \Pi_3: \end{matrix} \begin{cases} -9x + 6y + 15z = 1 \\ -5x + 8y + 9z = 3 \\ 4x + 2y - 6z = 0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} -9 & 6 & 15 & 1 \\ -5 & 8 & 9 & 3 \\ 4 & 2 & -6 & 0 \end{array}\right] \Leftrightarrow [A|B]$$

augmented form.

• $\left[\begin{array}{ccc|c} -9 & +6 & 15 & 1 \\ -5 & 8 & 9 & 3 \\ 4 & 2 & -6 & 0 \end{array}\right]$ $L_1 \leftarrow$ pivot line $L_1 \leftarrow -\frac{1}{9} L_1$

$$\left[\begin{array}{ccc|c} 1 & -6/9 & -15/9 & -1/9 \\ -5+5(1) & 8+5(-6/9) & 9+5(-15/9) & 3+5(-1/9) \\ 4-4(1) & 2-4(-6/9) & -6-4(-\frac{15}{9}) & 0-4(-\frac{1}{9}) \end{array}\right] \begin{matrix} L_1 \text{ pivot} \\ L_2 \leftarrow L_2 + 5L_1 \\ L_3 \leftarrow L_3 - 4L_1 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & -6/9 & -15/9 & -1/9 \\ 0 & \frac{72-30}{9} & \frac{81-75}{9} & \frac{27-5}{9} \\ 0 & \frac{18+24}{9} & \frac{-54+60}{9} & 4/9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -6/9 & -15/9 & -1/9 \\ 0 & 42/9 & 6/9 & 22/9 \\ 0 & 42/9 & 6/9 & 4/9 \end{array}\right] \begin{matrix} L_2 \leftarrow \text{pivot row} \\ L_3 \leftarrow L_3 - L_2 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & -6/9 & -15/9 & -1/9 \\ 0 & 42/9 & 6/9 & 22/9 \\ 0 & 0 & 0 & \frac{4-22}{9} \end{array}\right] L_2 \leftarrow \frac{9}{42} L_2$$

$$\left[\begin{array}{ccc|c} 1 & -6/9 & -15/9 & -1/9 \\ 0 & 1 & \frac{\cancel{6}}{\cancel{9}} \cdot \frac{\cancel{9}}{\cancel{42}} = 1/7 & \frac{\cancel{22}}{\cancel{9}} \cdot \frac{\cancel{9}}{\cancel{42}} = 11/21 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

To be in the ECHELON FORM the column of a pivot has all elements equal zero, but the pivot

$$\left[\begin{array}{ccc|c} 1+0 & -\frac{6}{9}+\frac{6}{9}(1) & -\frac{15}{9}+\frac{6}{9}\left(\frac{1}{7}\right) & -\frac{1}{9}+\frac{6}{9}\left(\frac{11}{21}\right) \\ 0 & 1 & 1/7 & 11/21 \end{array}\right] \begin{matrix} L_1 \leftarrow L_1 + \frac{6}{9} L_2 \\ L_2 \leftarrow \text{pivot row} \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1+0 & -\frac{6}{9}+\frac{6}{9}(1) & -\frac{15}{9}+\frac{6}{9}\left(\frac{1}{7}\right) & -\frac{1}{9}+\frac{6}{9}\left(\frac{11}{21}\right) \\ 0 & 1 & 1/7 & 11/21 \\ 0 & 0 & 0 & -2 \end{array}\right] \quad \begin{array}{l} L_1 \leftarrow L_1 + \frac{6}{9} L_2 \\ L_2 \leftarrow \text{pivot row} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{-7.15+6}{7.9} & \frac{-21+66}{9.21} \\ 0 & 1 & \frac{1}{7} & \frac{11}{21} \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -99/7.9 = -11/7 & 5/21 \\ 0 & 1 & 1/7 & 11/21 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

System has no solution..

Why? What is the geometric interpretation?

We have 3 planes (each row of the system is a plane)

What is their relative positions?

Let's compute $\Pi_1 \cap \Pi_3$ ^ $\Pi_2 \wedge \Pi_3$

$$\begin{array}{l} \Pi_1 \\ \Pi_3 \end{array} \begin{cases} -9x + 6y + 15z = 1 \\ 4x + 2y - 6z = 0 \quad \leftarrow \left(\frac{9}{4}\right) \end{cases}$$

$$+ \begin{cases} -9x + 6y + 15z = 1 \\ 9x + \frac{9}{2}y - 6.\frac{9}{4}z = 0 \end{cases}$$

$$0 \quad \left(6+\tfrac{9}{2}\right)y + \left(15 - 3.\tfrac{9}{2}\right)z = 1 + 0$$

$$\left(\frac{12+9}{2}\right)y + \left(\frac{30-27}{2}\right)z = 1$$

$$\frac{21}{2}y + \frac{3}{2}z = 1 \Rightarrow y = \frac{2}{21} - \cancel{\frac{2}{21}} \cdot \cancel{\frac{3}{2}} z$$

$$\boxed{y = \frac{2}{21} - \frac{1}{7}z} \quad (a)$$

we substitute (a) in equation of Π_1.

$$-9x + 6\left(\frac{2}{21} - \frac{1}{7}z\right) + 15z = 1$$

$$-9x + \frac{12}{21} - \frac{6}{7}z + 15z = 1$$

$$-9x + \frac{12}{21} - \frac{6}{7}z + 15z = 1$$

$$-9x = 1 - \frac{12}{21} + \left(\frac{6}{7} - 15\right)z$$

$$-9x = \frac{21-12}{21} + \left(\frac{6 - 15\times 7}{7}\right)z$$

$$-9x = \frac{9}{21} - \frac{99}{7}z \Rightarrow \boxed{x = -\frac{1}{21} + \frac{11}{7}z}$$

So the solution of $\pi_1 \cap \pi_3$ is:

$$\begin{aligned} x &= -\frac{1}{21} + \frac{11}{7}z \\ y &= \frac{2}{21} - \frac{1}{7}z \\ z &= 0 + 1z \end{aligned} \quad \xRightarrow[\lambda \in \mathbb{R}]{z = \lambda} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/21 \\ 2/21 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 11/7 \\ -1/7 \\ 1 \end{pmatrix}$$

line passing through $\left(-\frac{1}{21}, \frac{2}{21}, 0\right)$
with direction vector $\vec{v} = \left(\frac{11}{7}, -\frac{1}{7}, 1\right)$

<u>Now</u> Compute $\pi_2 \cap \pi_3$

$$+\begin{cases} -5x + 8y + 9z = 3 \\ 4x + 2y - 6z = 0 \end{cases} \qquad L_3 \leftarrow \frac{5}{4} L_3$$

$$\left(-5 + \frac{5}{4}(4)\right)x + \left(8 + \frac{5}{\cancel{4}_2}\cancel{2}\right)y + \left(9 + \frac{5}{\cancel{4}_2}(-\cancel{6}^{3})\right)z = 3$$

$$\left(\frac{16+5}{2}\right)y + \left(\frac{18-15}{2}\right)z = 3$$

$$\frac{21}{2}y + \frac{3}{2}z = 3$$

$$y = \cancel{3}\cdot\frac{2}{\cancel{21}} - \frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{21}}z$$

$$\boxed{y = \frac{2}{7} - \frac{1}{7}z} \quad (a)$$

Substituing (a) on the eq of π_2

$$-5x + 8\left(\frac{2}{7} - \frac{1}{7}z\right) + 9z = 3$$

$$5x \quad 1 - \frac{16}{7} - \frac{8}{7}z + 9z = 3$$

$$-5x = 3 - \frac{16}{7} + \left(-9 + \frac{8}{7}\right)z$$

$$-5x = \frac{5}{7} + \left(\frac{-63+8}{7}\right)z$$

$$-5x = \frac{5}{7} - \frac{55}{7}z \Rightarrow \boxed{x = -\frac{1}{7} + \frac{11}{7}z}$$

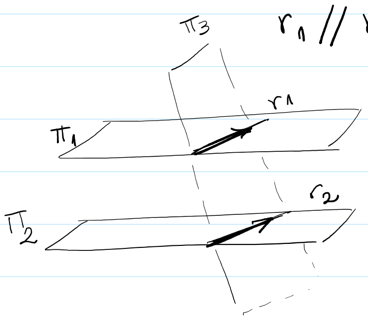
$$-5x = \frac{5}{7} - \frac{55}{7}z \Rightarrow \boxed{x = -\frac{1}{7} + \frac{11}{7}z}$$

So $\pi_2 \cap \pi_3$ is a line of equation

$$\begin{aligned} x &= -1/7 + 11/7\, z \\ y &= 2/7 - 1/7\, z \\ z &= 0 + 1z \end{aligned} \quad \xRightarrow[\beta\in\mathbb{R}]{z=\lambda} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/7 \\ 2/7 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 11/7 \\ -1/7 \\ 1 \end{pmatrix}$$

<u>Observe</u>: $\pi_1 \cap \pi_3 = r_1 : (x) = \begin{pmatrix} -1/21 \\ 2/21 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 11/7 \\ -1/7 \\ 1 \end{pmatrix}$

$\pi_2 \cap \pi_3 = r_2 : (x) = \begin{pmatrix} -1/7 \\ 2/7 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 11/7 \\ -1/7 \\ 1 \end{pmatrix}$



$r_1 // r_2 \qquad r_1 \cap r_2 = \emptyset$

$\bar{v} = (11/7, -1/7, 1)$

$-\bar{v} = (-11/7, 1/7, -1)$ is also a direction vector of r_1 and r_2

In fact, any multiple of $\bar{v}$ is a direction vector of the two lines.

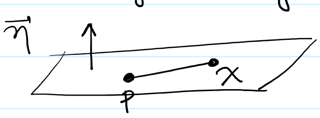
But now, just come back to the echelon form of the augmented matrix ... and look what is appearing on the third column.. More about that comes in <u>MVE465</u>!

Adams Bok.

Now we are focusing on the exercises of § 10.4 pages 599 & 600.

Exercises <u>2 - 9</u>: Find the eq. of the plane given the conditions:

2) Passing throug $P = (0, 2, -3)$ and perpendicular to $(4, 1, -2) = \vec{n}$



$\forall$ point $X \in \pi$, we have

$\overline{PX} \perp \vec{n} \Rightarrow \overline{PX} \cdot \vec{n} = 0$

$PX = (x, y, z) - (0, 2, -3) = (x - 0, y - 2, z + 3)$

$PX \cdot \vec{n} = (x - 0, y - 2, z + 3) \cdot (4, 1, -2) = 0 \Rightarrow$

$4x + 1y - 2z - (\cancel{4 \cdot 0} + 1 \cdot 2 + 6) = 0$

$\overline{PX} \bullet \eta = (x-0, y-2, z+3)\cdot(4,1,-2) = 0 \Rightarrow$

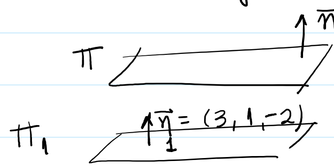
$4x + 1y - 2z - (\cancel{4.0} + 1.2 + 6) = 0$

$\Pi: \quad 4x + y - 2z = 8$

3) Passing through the origin $O = (0,0,0)$ & $\overrightarrow{\eta} = (1,-1,2)$

<u>Same idea.</u>

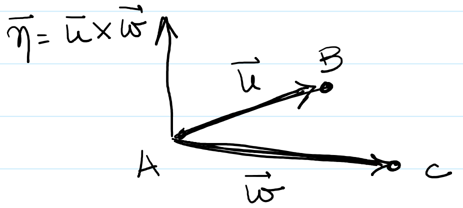
4) Passing through $P = (1,2,3)$ and parallel to $\Pi_1: 3x + y - 2z = 15$



$\pi \parallel \pi_1$ so $\overrightarrow{\eta_\pi} \parallel \overrightarrow{\eta_1}$. $\overrightarrow{\eta_\pi} = (3,1,-2)$

Knowing P & $\overrightarrow{\eta_\pi}$ it follows as before: $\overline{PX} \bullet \overline{\eta_\pi} = 0$

6) Passing throug 3 points $A = (-2,0,0)$, $B = (0,3,0)$ $C = (0,0,4)$



$\overline{AB} = \vec{u}$ $\qquad$ $\overline{u} \times \overline{w} = \overline{\eta}$

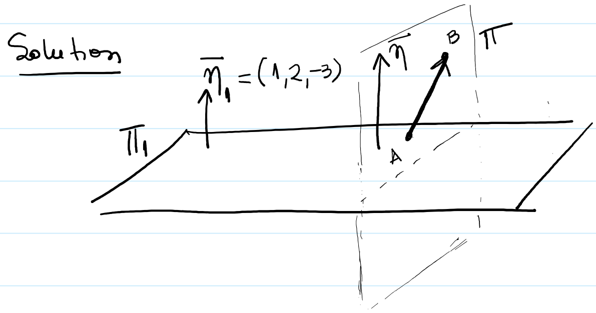
$\overline{AC} = \vec{w}$

$P =$ can be any one of the 3 given

<u>Obs</u>: Ex 5) is analogous.

7) Passing through $A = (1,1,1)$ & $B = (2,0,3)$ and perpendicular to the plane $\pi_1: x + 2y - 3z = 0$

<u>Solution</u>



π is the plane we are looking for.

π has $\overline{\eta_1} = (1,2,-3)$ &

$\overline{AB} = (2,0,3) - (1,1,1) = (1,-1,2)$

as its direction vectors:

so $\overline{\eta_\pi} = \overline{\eta_1} \times \overline{AB}$ and $P = A$ or B.

$$\overline{\eta_\pi} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -3 \\ 1 & -1 & 2 \end{vmatrix} = \vec{i}\underbrace{[4-3]}_{=1} - \vec{j}\underbrace{[2+3]}_{5} + \vec{k}[-1-2]$$

$\vec{\eta}_\pi = (1, -5, -3)$

$\overline{AX} \cdot \overline{\eta}_\pi = 0 \Rightarrow (x-1, y-1, z-1) \cdot (1, -5, -3) = 0$

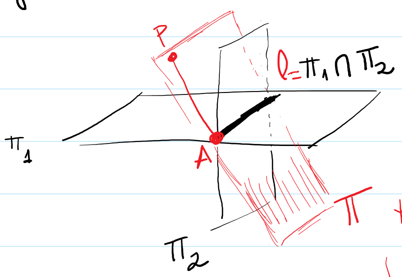
$x - 1 - 5y + 5 - 3z + 3 = 0$

$x - 5y - 3z - 1 + 5 + 3 = 0$

$\boxed{x - 5y - 3z = -7}$ π.

8. Passing through the line of intersection of the Planes:
π_1: $2x + 3y - z = 0$, π_2: $x - 4y + 2z = -5$ and
passing through $P = (-2, 0, 1)$

Solution



$\begin{cases} A = \text{point of the line } l. \\ \overline{v} = \text{director vector of } l \end{cases}$

π that we are looking for.

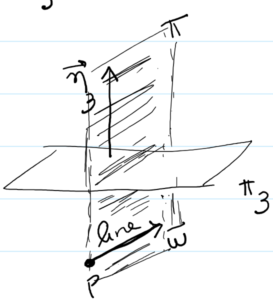
$\begin{cases} \overline{AP} \times \overline{v} = \overline{\eta}_\pi \\ P \text{ is given} \end{cases}$ so $\underbrace{(XP) \cdot \overline{\eta}_\pi = 0}_{\text{equation of } \pi}$

9. Passing through the line $x + y = 2$, $y - z = 3$
and perpendicular to the plane $2x + 3y + 4z = 5$

Solution:

line $\pi_1 \cap \pi_2$ $\begin{cases} x + y + 0z = 2 \\ 0 + y - z = 3 \end{cases}$ $\Rightarrow$ this will provide $P \in \pi$
and one of the vectors to generate π: $\overline{w}$

π_3: $2x + 3y + 4z = 5$ $\Rightarrow$ Since $\pi \perp \pi_3$, $\vec{\eta}_3 // \pi$ $\therefore$ $\overline{\eta}_3$ provides
$\vec{\eta}_3 = (2, 3, 4)$ the second generator we need to obtain $\overline{\eta}_\pi$
$\vec{\eta}_\pi = \overline{w} \times \overline{\eta}_3$.



So: line: $\begin{cases} x + y = 2 \\ y - z = 3 \Rightarrow y = 3 + z \end{cases}$

$x + (3 + z) = 2 \Rightarrow x = 2 - 3 - z$

$\boxed{x = -1 - z}$

$$\begin{cases} x = -1 - z \\ y = 3 + z \\ z = z \end{cases} \Rightarrow z = \lambda \in \mathbb{R} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}}_{P \in \pi} + \lambda \underbrace{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}_{\vec{w} \,//\, \pi}$$

Now compute $\vec{n}_\pi = \vec{w} \times \vec{n}_3 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} =$

$= \vec{i}(4-3) - \vec{j}(-4-2) + \vec{k}(-3-2) = 1\vec{i} + 6\vec{j} - 5\vec{k} = (1,6,-5)$

$\left.\begin{array}{l} \vec{n}_\pi = (1,6,-5) \\ P = (-1,3,0) \end{array}\right\}$ $\vec{PX} \perp \vec{n}_\pi \therefore \vec{PX} \bullet \vec{n}_\pi = 0 \Rightarrow$

$(x-(-1), y-3, z-0) \bullet (1,6,-5) = 0 \Rightarrow$

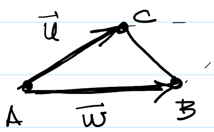
$x + 1 + 6y - 18 - 5z = 0$

$x + 6y - 5z = 18 - 1$

$$\boxed{\pi: \ x + 6y - 5z = 17}$$

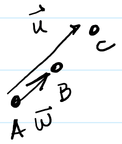
10. → Under what geometric condition will 3 distinct points in $\mathbb{R}^3$ not determine a unique plane passing through them?
→ How can be this condition be expressed algebraically in terms of the position of the position vectors?

If A, B, C are



Area $\neq 0 = |\vec{u} \times \vec{w}| \Rightarrow$
$\Rightarrow \exists\, \pi_{A,BC}$

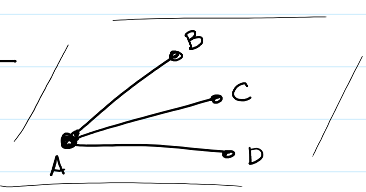
If A, B, C are



Area $= 0 = |\vec{u} \times \vec{w}| \Rightarrow$
$\Rightarrow \nexists\, \pi_{ABC}$, $\exists$ line$_{ABC}$

11) Give a condition on the position vectors of four points that guarantees that the four points are coplanar.

Solution



there 3 vectors $\vec{AB}$, $\vec{AC}$ & $\vec{AD}$ do not generate a parallelepiped. ⇒ there is no volume. ⇒

$\vec{AB} \bullet \vec{AC} \times \vec{AD} = 0$

12) $x + y + z = \lambda$, $\lambda \in \mathbb{R}$ family of parallel planes.

$\vec{n} = (1,1,1)$ $\uparrow \vec{n} = (1,1,1)$

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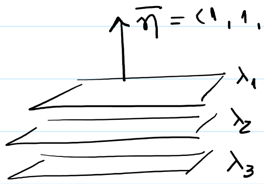
$\vec{n} = (1,1,1)$



13) $\underbrace{x + \lambda y + \lambda z}_{} = \lambda$

$a = 1$
$b = \lambda$
$c = \lambda$

$(x-x_0, y-y_0, z-z_0) \cdot (a,b,c) = 0$

$ax - ax_0 + by - by_0 + cz - cz_0 = 0$

$\underbrace{ax + by + cz}_{} = \underbrace{ax_0 + by_0 + cz_0}_{d}$

$\Rightarrow \quad d = \lambda \Rightarrow \quad 1x_0 + \lambda y_0 + \lambda z_0 = \lambda$

$x_0 + \lambda(\underbrace{y_0 + z_0}_{}) = 1 \cdot \lambda + 0 \quad \Rightarrow x_0 = 0$

$\quad \& \quad y_0 + z_0 = 1 \quad \Rightarrow y_0 = 1 - z_0$

So, we have a family of planes with

$\vec{n} = (1, \lambda, \lambda)$ & $P = (0, 1 - z_0, z_0)$ $P: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ (*)

The points on the plane are chosen as being from line (*).

Equations of lines   Exercises 15 - 19

15) Passing through $P = (1,2,3)$ & parallel to the direction $\vec{v} = (2,-3,4)$

Solution: $P \in$ line

$\vec{v} =$ direction vector of line

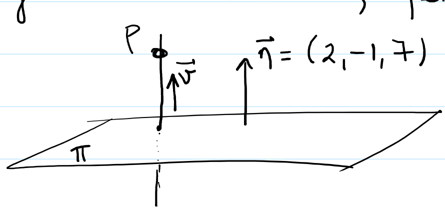
1) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$, $\lambda \in \mathbb{R}$.   vectorial form

2) $\begin{cases} x = 1 + 2\lambda \\ y = 2 - 3\lambda \\ z = 3 + 4\lambda \end{cases}$ parametric form

3) $\dfrac{x-1}{2} = \dfrac{y-2}{-3} = \dfrac{z-3}{4}$ standard form

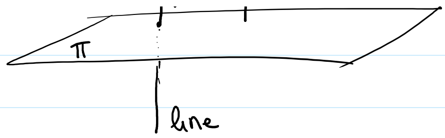
16) Passing through $P = (-1, 0, 1)$, perpendicular to $\Pi: 2x - y + 7z = 12$

Solution



$\vec{v} \parallel \vec{n} \quad \therefore \quad \vec{v} = \vec{n}$ is one choice

$\boxed{X = P + \lambda \vec{v}}$ line



$\boxed{X = P + \lambda \vec{v}}$ line

17) Through $O = (0,0,0)$ & parallel to $l_1 : \Pi_1 \cap \Pi_2$ $\begin{cases} x + 2y - z = 2 \\ 2x - y + 4z = 5 \end{cases}$

Solution:

→ find l_1 and obtain its direction vector.

Since the line we are looking for is parallel to l_1, its direction vector is a multiple of the direction vector of l_1.

→ $P = O$

18) Find the line l that passes through $P = (2,-1,1)$ & is parallel to each of the two planes $\Pi_1 : x + y = 0$ & $\Pi_2 : x - y + 2z = 0$.

Solution: P of l is given

$\vec{v}$ = direction vector of l = ?

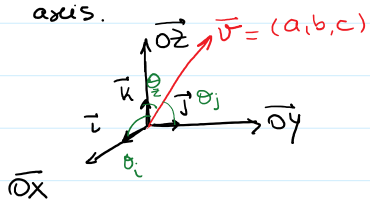
Since Π_1 is not parallel to Π_2,
there is only one way that l can be parallel to Π_1 & Π_2 at the same time, $l \,//\, l_1 = \Pi_1 \cap \Pi_2$.

So: find $l_1 = \Pi_1 \cap \Pi_2$, obtain $\vec{w}$ = the direction vector from l_1

& build l equation with $P = (2,-1,1)$ & $\vec{w}$.

19. l is the line passing through $P = (1,2,-1)$ & making equal angles with the positive directions of the coordinate axis.

Solution:



$a = \vec{v} \cdot \vec{i} = \|\vec{v}\| \|\vec{i}\| \cos\theta_i$

$b = \vec{v} \cdot \vec{j} = \|\vec{v}\| \|\vec{j}\| \cos\theta_j$

$c = \vec{v} \cdot \vec{k} = \|\vec{v}\| \|\vec{k}\| \cos\theta_z$

$\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$

By hypothesis, $\cos\theta_i = \cos\theta_j = \cos\theta_z$ $\Rightarrow$ $a = b = c = \underbrace{\|\vec{v}\| \cos\theta}_{\text{constant}} = \alpha$

So $\vec{v} = (a,b,c) = (\alpha, \alpha, \alpha) = \alpha(1, 1, 1)$

So $l : X = P + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

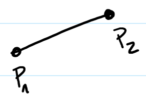
23) If $P_1 = (x_1, y_1, z_1)$ & $P_2 = (x_2, y_2, z_2)$, show that the equations

23) If $P_1=(x_1,y_1,z_1)$ & $P_2=(x_2,y_2,z_2)$, show that the equations

$$(A)\begin{cases} x = x_1 + t\,(\overbrace{x_2-x_1}^{a}) \\ y = y_1 + t\,(\overbrace{y_2-y_1}^{b}) \\ z = z_1 + t\,(\overbrace{z_2-z_1}^{c}) \end{cases}$$

represent the line through P_2 & P_1.

Solution.

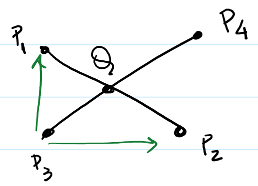


$$\overrightarrow{P_1P_2} = P_2 - P_1 = (x_2,y_2,z_2)-(x_1,y_1,z_1) =$$
$$= (\underbrace{x_2-x_1}_{a}, \underbrace{y_2-y_1}_{b}, \underbrace{z_2-z_1}_{c})$$

So (A) is the scalar parametric equations of the line passing through P_1 with direction vector $\overrightarrow{P_1P_2}=(a,b,c)$.

25) Under what conditions on the position vectors of 4 points P_1, P_2, P_3 & P_4 will the straight line through P_1 and P_2 intersect the straight line through P_3 and P_4 at a unique point?

Solution



$$\begin{cases} \overrightarrow{P_1P_2} \text{ can not be parallel to } \overrightarrow{P_3P_4} \quad (\overrightarrow{P_1P_2} \not\parallel \overrightarrow{P_3P_4}) \\ P_1P_2P_3P_4 \text{ must be on the same plane } \therefore \end{cases}$$

They don't form a parallelepiped.

So $\overrightarrow{P_3P_1}$, $\overrightarrow{P_3P_4}$ & $\overrightarrow{P_3P_2}$ generate no 3-D geometric object. So

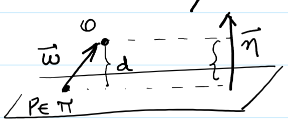
$$\overrightarrow{P_3P_1} \cdot \overrightarrow{P_3P_4} \times \overrightarrow{P_3P_2} = 0.$$

Find Distances   Exercises 26 – 29.

26. From $O=(0,0,0)$ to $\Pi: x+2y+3z=4$

Solution: 1) $O \notin \Pi$

2)



$w=\overrightarrow{PO}$

$$d = \|\vec{\omega}_{\vec{\eta}}\| = \left\|\operatorname{proj}_{\vec{\eta}}\vec{\omega}\right\| = \frac{|\vec{\omega}\cdot\vec{\eta}|}{\vec{\eta}\cdot\vec{\eta}}\,\|\vec{\eta}\|$$

$$d = \frac{|\vec{\omega}\cdot\vec{\eta}|}{\|\vec{\eta}\|^{\cancel{2}}}\cdot\cancel{\|\vec{\eta}\|} = \frac{|\vec{\omega}\cdot\vec{\eta}|}{\|\vec{\eta}\|}$$

$$d = \frac{|(0,0,4/3)\cdot(1,2,3)|}{\sqrt{1+4+9}} = \frac{4}{\sqrt{14}}$$

28. $O=(0,0,0)$ $\ell = \pi_1 \cap \pi_2$ $d(O,\ell) = ?$

$$\begin{matrix}\pi_1: \\ \pi_2: +\end{matrix}\begin{cases} x+y+z=0 \\ 2x-y-5z=1 \end{cases}$$

Substitute (a) in Π_1-equation

$$\left(\tfrac{1}{3}+\tfrac{4}{3}z\right)+y+z=0$$

$$\begin{cases} \pi_1: & x+y+z=0 \\ \pi_2: & 2x-y-5z=1 \end{cases} \quad +$$

$$3x \qquad -4z = 1$$

$$3x = 1+4z$$

$$\text{(a)} \quad \boxed{x = \tfrac{1}{3} + \tfrac{4}{3}z}$$

Substitute (a) in π_1-equation

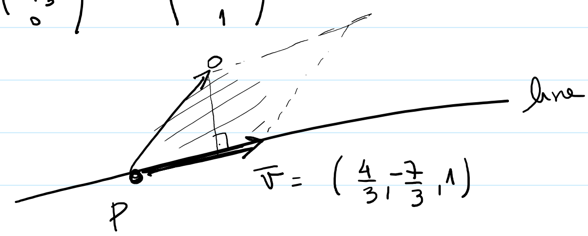
$$\left(\tfrac{1}{3} + \tfrac{4}{3}z\right) + y + z = 0$$

$$y = -\tfrac{1}{3} + \left(-\tfrac{4}{3} - 1\right)z$$

$$\boxed{y = -\tfrac{1}{3} - \tfrac{7}{3}z}$$

$$x = \tfrac{1}{3} + \tfrac{4}{3}z$$
$$y = -\tfrac{1}{3} - \tfrac{7}{3}z$$
$$z = z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4/3 \\ -7/3 \\ 1 \end{pmatrix}; \ \lambda \in \mathbb{R}.$$



Area $= b * h \implies$

$$h = \frac{\text{Area}}{b} = \frac{\|\vec{OP} \times \vec{v}\|}{\|\vec{v}\|}$$

Extra exercises

Given the sphere $S: x^2 - 2x + y^2 - 4y + z^2 - 2z = -2$

- Find the two tangent planes to S which are parallel to $\pi: x+y+z=4$.
- Find also the two tangent points T_1 & T_2
- Find the equation of the line which passes through T_1 & T_2
- Does the center of S belong to this line?

Solution: 1) Find the center of S & its radius, by recovering the binomial terms.

2) localize π with respect to S

1) $\underbrace{x^2 - 2x} + \underbrace{y^2 - 4y} + z^2 - 2z = -2$

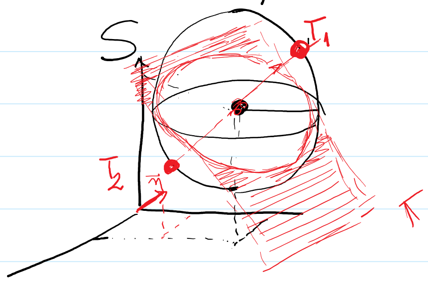
$$x^2 - 2\square x + \square^2 - \square^2 + y^2 - 2\heartsuit y + \heartsuit^2 - \heartsuit^2 + z^2 - 2\clubsuit z + \clubsuit^2 - \clubsuit^2 = -2$$

$$\cancel{-}\cancel{2}\square\cancel{x} = \cancel{-}\cancel{2}\cancel{x} \qquad \square = 1 \Rightarrow \square^2 = 1 \qquad \cancel{-}2\heartsuit y = \cancel{-}4y \Rightarrow 2\heartsuit = 4 \quad \heartsuit = 2 \Rightarrow \heartsuit^2 = 4$$

$$\underbrace{x^2 - 2x + 1}_{(x-1)^2} - 1 + \underbrace{y^2 - 4y + 4}_{(y-2)^2} - 4 + \underbrace{z^2 - 2z + 1}_{(z-1)^2} - 1 = -2$$

$$(x-1)^2 + (y-2)^2 + (z-1)^2 = \cancel{-2} \cancel{+1} + 4 \cancel{+1}$$

$$S: (x-1)^2 + (y-2)^2 + (z-1)^2 = 4 \qquad \begin{cases} C = (1, 2, 1) \\ r = 2 \end{cases}$$



$\Pi: x+y+z=4$? $G \in \Pi$? $(1)+(2)+(1)=4$?

yes! $G \in \Pi$.

$\vec{\eta}_\pi = (1,1,1)$

3) To find T_1 & T_2, first we need to find the eq of the line passing through G & $// \vec{\eta} = (1,1,1)$

$$l: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+\lambda 1 \\ 2+\lambda 1 \\ 1+\lambda 1 \end{pmatrix}$$

4) Now we must compute $l \cap S$.

This means, we substitute l equation in S equation.

$$S: (x-1)^2 + (y-2)^2 + (z-1)^2 = 4$$

$$(\cancel{1}+\lambda-\cancel{1})^2 + (\cancel{2}+\lambda-\cancel{2})^2 + (\cancel{1}+\lambda-\cancel{1})^2 = 4$$

$$\lambda^2 + \lambda^2 + \lambda^2 = 4 \Rightarrow \lambda^2 = 4/3 \quad \lambda = \pm 2/\sqrt{3}$$

5) T_1 is then obtained from l equation assuming $\lambda = 2/\sqrt{3}$

T_2 is obtained assuming $\lambda = -2/\sqrt{3}$

$$T_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 2/\sqrt{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2/\sqrt{3} \\ 2+2/\sqrt{3} \\ 1+2/\sqrt{3} \end{pmatrix}$$

$$T_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - 2/\sqrt{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-2/\sqrt{3} \\ 2-2/\sqrt{3} \\ 1-2/\sqrt{3} \end{pmatrix}$$

So $\Pi_{T_1}: (\overrightarrow{XT_1}) \cdot \vec{\eta} = 0 \quad \therefore \quad x+y+z = -4 - \frac{6}{\sqrt{3}}$

$\Pi_{T_2}: (\overrightarrow{XT_2}) \cdot \vec{\eta} = 0 \quad \therefore \quad x+y+z = -4 + 6/\sqrt{3}$

Important exercises from <u>§10.3</u> page 592 Adams Bok.

13. If $\underbrace{\vec{u}+\vec{v}+\vec{w}=\vec{0}}_{\circledast}$, show that $\vec{u}\times\vec{v} = \vec{v}\times\vec{w} = \vec{w}\times\vec{u}$

<u>Solution.</u>

a) $\circledast \quad \vec{u} = -\vec{v} - \vec{w}$

$$\vec{0} = \vec{u} \times \vec{u} \Rightarrow (-\vec{v} - \vec{w}) \times \vec{u} = \vec{0} \Rightarrow \vec{u} \times (\vec{v} + \vec{w}) = \vec{0} \Rightarrow$$

$$\Rightarrow \vec{u} \times \vec{v} + \vec{u} \times \vec{w} = 0 \Rightarrow \underline{\vec{u} \times \vec{v}} = -\vec{u} \times \vec{w} = \underline{\vec{w} \times \vec{u}}$$

b) $\vec{w} = -(\vec{u} + \vec{v})$

$$\vec{0} = \vec{w} \times \vec{w} = -(\vec{u} + \vec{v}) \times \vec{w} = -\vec{u} \times \vec{w} - \vec{v} \times \vec{w} \Rightarrow$$

$$\vec{u} \times \vec{w} + \vec{v} \times \vec{w} = \vec{0} \Rightarrow \vec{u} \times \vec{w} = -\vec{v} \times \vec{w} \Rightarrow$$

$$\cancel{-}\, \vec{w} \times \vec{u} = \cancel{-}\, \vec{v} \times \vec{w}$$

Therefore $\boxed{\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}}$

14) Volume of a Tethrahedron $\equiv$ pyramid with a triangular base & three other triangular faces. $V = \frac{1}{3} \Delta . h$

Show that: the volume of a thetrahedron generated by 3 vectors is one-sixth ($1/6$) of the volume of the parallelepiped spanned by the same vectors

<u>Solution:</u> $\vec{u}, \vec{v}, \vec{w}$ three non coplanar vectors.

$$V_{\text{parallelepiped}} = |\vec{u}|\,|\vec{v} \times \vec{w}| \cos\theta = \underbrace{|\vec{u}| \cos\theta}_{h} \cdot \underbrace{|\vec{v} \times \vec{w}|}_{\text{Area of the basis}} =$$

$$= |\vec{u} \cdot \vec{v} \times \vec{w}|$$

Now for the thetrahedron, the base is no longer the parallelogram formed by $\vec{v}$ & $\vec{w}$.
It is half of it.

So now Area of base $= \frac{1}{2} |\vec{v} \times \vec{w}|$

So the Volume of the Thetrahedron is now

$$V_{\Delta} = \frac{1}{3} \left(\frac{1}{2} |\vec{v} \times \vec{w}| \right) \cdot \underbrace{|\vec{u}| \cos\theta}_{\text{the same height.}}$$

$$V_{\Delta} = \frac{1}{6} |\vec{u}|\,||\vec{v} \times \vec{w}||\cos\theta = \frac{1}{6} |\vec{u} \cdot \vec{v} \times \vec{w}| = \frac{1}{6} V_{\text{paralle.}}$$

Lycka till!