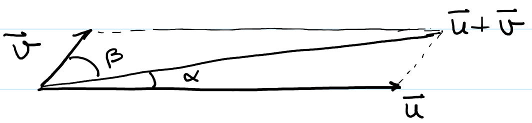
Exercise25_page584_section10.2

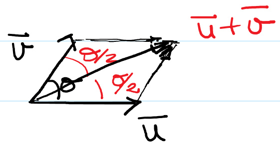
den 16 oktober 2020 14:15

[Find a unit vector that bisects the angle between any two nonzero vectors $\bar{u}$ & $\bar{v}$.]

Ex 25
page 584
§ 10.2.

Solution:





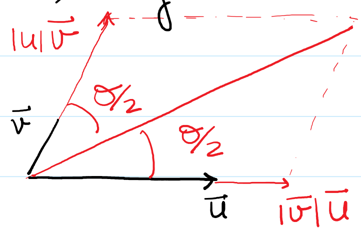
$\alpha+\beta=\theta$

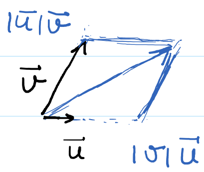
since $|\bar{u}| \neq |\bar{v}|$ $\quad \alpha \neq \theta/2$

$\beta \neq \theta/2$

$|\bar{u}| = |\bar{v}| \Rightarrow \vec{u}+\vec{v}$ is a vector whose direction is given by the bissection of θ, which is the angle between $\vec{u}$ and $\vec{v}$

So given any two non parallel vectors with different sizes it is possible to obtain two other with the same size, preserving the original directions.





$|\bar{u}| > |\bar{v}|$

But now $|v|\,\bar{u}$ has the same size as $|\bar{u}|\,\bar{v}$

$|\bar{u}| < |\bar{v}|$

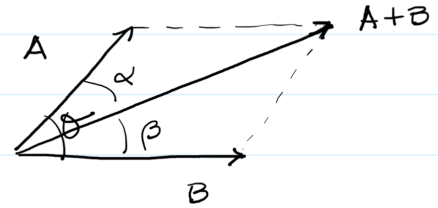
But now $|v|\,\bar{u}$ has the same size as $|\bar{u}|\,\bar{v}$

$$\left.\begin{aligned} \bar{A} = |\bar{u}|\,\bar{v} &\Rightarrow |\bar{A}| = |\bar{u}|\,|\bar{v}| \\ B = |\bar{v}|\,\bar{u} &\Rightarrow |\bar{B}| = |\bar{v}|\,|\bar{u}| \end{aligned}\right] = \text{these values are equal.}$$

So given any two non parallel vectors $\bar{u}$ & $\bar{v}$ we consider $\bar{A} = |\bar{u}|\,\bar{v}$ & $\bar{B} = |\bar{v}|\,\vec{u}$, both with the same size.

Now we are going to show that the angle α between $\bar{A}+\bar{B}$ & $\bar{A}$ is equal to the angle β between $\vec{A}+\vec{B}$ & $\vec{B}$.

(So $\alpha = \beta = \frac{\theta}{2}$, θ being the angle between $\vec{A}$ & $\vec{B}$



$\alpha = \measuredangle(\vec{A}+\vec{B}, \vec{A})$

$\beta = \measuredangle(\vec{A}+\vec{B}, \vec{B})$

$\theta = \measuredangle(\vec{A}, \vec{B})$

$|\vec{A}| = |\vec{B}| = K$

(*) So, on one hand we have

$(\vec{A}+\vec{B}) \cdot \vec{A} = \vec{A}\cdot\vec{A} + \vec{B}\cdot A = K^2 + \vec{B}\cdot\vec{A}$

$(\vec{A}+\vec{B}) \cdot \vec{B} = \vec{A}\vec{B} + B\cdot B = K^2 + \vec{A}\cdot\vec{B}$

Recall

$A \cdot A = \|A\|^2 = K^2$

$B \cdot B = \|B\|^2 = K^2$

$\boxtimes$ On the other hand

$(\vec{A}+\vec{B}) \cdot \vec{A} = \|A+B\| \cdot \|A\| \cos(\alpha) \Rightarrow \|A+B\| . K \cos\alpha$

$(\vec{A}+\vec{B}) \cdot \vec{B} = \|A+B\| \, \|B\| \cos(\beta) \Rightarrow \|A+B\| . K \cos\beta$

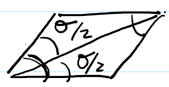
(*) Since $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$, we have $(A+B)\cdot A = (A+B)\cdot B$

Thus, using $\boxtimes$, we have

$\cancel{\|A+B\|}\cancel{K}\cos\alpha = (\vec{A}+\vec{B})\vec{A} = K^2 + \vec{B}\cdot\vec{A} = (A+B)\cdot B = \cancel{\|A+B\|}\cancel{K}\cos\beta$

$\Rightarrow \boxed{\cos\alpha = \cos\beta}$

Since we have a parallelogram with all sides with the same length, by symmetry $\alpha = \beta = \theta/2$



So the vector $\vec{A} + \vec{B} = |\vec{u}|\vec{v} + |\vec{v}|\vec{u}$ has the property we wanted.

But we still need to let it with size 1.

So $\frac{A+B}{\|A+B\|}$ is the unitary vector we are

So $\frac{A+B}{\|A+B\|}$ is the unitary vector we are looking for.

$$\frac{|\vec{u}|\,\vec{v} + |\vec{v}|\,\vec{u}}{\|\,|\vec{u}|\,\vec{v} + |\vec{v}|\,\vec{u}\,\|}$$

$\square$.