Föreläsning 7.3

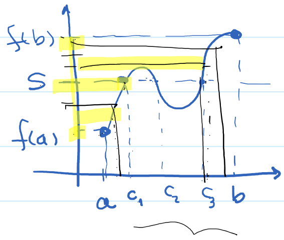
den 14 oktober 2020 16:21

Repetition 1. Vi ska arbeta med Bevislista och fokusera på frågor som ni valt i enkät.

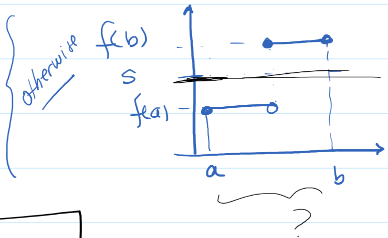
1) Medelvärdessatsen ≡ The Mean-Value Theorem page 138 Adams

* Before presenting this theorem, I would like to recall
The Intermediate-Value Theorem page 85 Adams

If [$f(x)$ is continuous on the Interval $I=[a,b]$ & $f(a) < s < f(b)$] Then [There exists a number $c \in [a,b]$, such that $f(c) = s$]



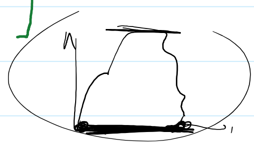
f has to be Continuous.



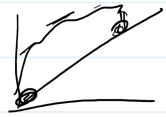
$f(a) < s < f(b)$
BUT $\nexists$ $c \in [a,b]$ such that $f(c) = s$.

Theorem 15: The Rolle's Theorem $\boxed{\text{page 142}}$

If [g continuous on $[a,b]$ & g differentiable on (a,b) & $\boxed{g(a) = g(b)}$] Then [$\exists\, c \in (a,b)$ (there exists c in (a,b)) such that $g'(c) = 0$]



Proof of the Rolle's Theorem. (page 143)



1^{st} case: { If $g(x) = g(a)$, $\forall x \in [a,b]$, Then $g(x) \equiv$ constant function
And so $g'(c) = 0$, $\forall\, c \in (a,b)$.

2^{nd} case: $g(x) \neq$ constant function.
So, there must exist $x \in (a,b)$ such that
$g(x) \neq g(a)$ (since $g(a) = g(b)$, $g(x) \neq g(b)$ also)

$\boxed{\text{Let's assume } g(x) > g(a)}$

By the Max-Min Theorem (Theo 8, Section 1.4), being continuous on $[a,b]$, g must have a Max value at some point $c \in [a,b]$. So
$g(c) \geq g(x) > g(a) = g(b)$ c cannot be equal to either a or b.

Therefore $c \in (a,b)$ some point inside the open interval
So, by hypothesis g is differentiable at $c \in (a,b)$.

Therefore $c \in (a,b)$ some point inside the open interval

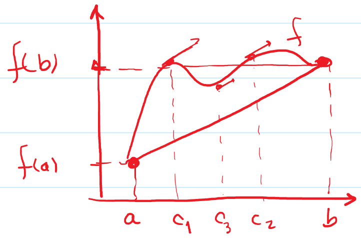
So, by hypothesis g is differentiable at $c \in (a,b)$.

(*) By Theorem 14, c must be a critical point of g:

So $g'(c) = 0$ ■

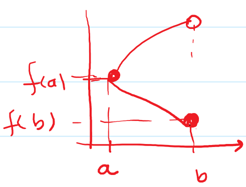
The Mean-Value Theorem page 138 (Theorem 11)

If $\left[\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{array}\right]$ Then $\left[\begin{array}{l} \text{There exists } c \in (a,b) \text{, such that} \\ \dfrac{f(b)-f(a)}{b-a} = f'(c) \end{array}\right]$

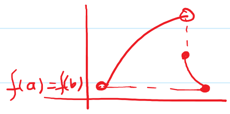


Obs f must be continuous on $[a,b]$
f must be differentiable on (a,b),

otherwise:



$f(x)$ is discontinuous.



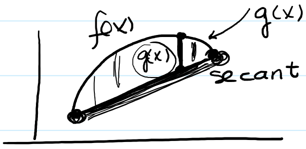
f is NOT Differentiable

Proof of the Mean-Value Theorem (page 143.)

Suppose that f satisfies the conditions of the Theorem $\left(\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{array}\right)$

Let's construct an auxiliar function $g(x)$ such that:

$$g(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right)$$



For $a \leq x \leq b$, $g(x)$ is the vertical displacement between the curve $y = f(x)$ and the chord line $y = f(a) + \dfrac{(f(b)-f(a))}{(b-a)}(x-a)$

(i) The function g is also continuous & differentiable on (a,b)
Because g is the subtraction of 2 functions with the same properties.

(ii) $\begin{cases} g(a) = f(a) - f(a) + 0 = \boxed{0} \\ g(b) = f(b) - f(a) - \dfrac{(f(b)-f(a))}{\cancel{(b-a)}}\cancel{(b-a)} = f(b) - f(a) - (f(b) - f(a)) = 0 \end{cases}$

So $g(a) = 0$ & $g(b) = 0$ & g not being a constant function on (a,b),
By the Rolle's Theorem, $\exists\ c \in (a,b)$ such that $g'(c) = 0$

$$g'(x) = f'(x) - \left(\frac{f(b)-f(a)}{b-a}\right) \cdot 1$$

$$g'(x) = f'(x) - \left(\frac{f(b)-f(a)}{b-a}\right)\cdot 1$$

$$g'(c) = 0 \Rightarrow f'(c) - \left(\frac{f(b)-f(a)}{b-a}\right) = 0 \Rightarrow \boxed{f'(c) = \left(\frac{f(b)-f(a)}{b-a}\right)}$$

Theorem 16 page 144 : The Generalized Mean-Value Theorem

If [functions f & g are continuous on $[a,b]$ & differentiable on (a,b)] & [$g'(x) \neq 0, \forall x \in (a,b)$] Then [there exists a number $c \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$]

Obs: it is the same value c for both!

Proof of the Generalized Mean-Value Theorem.

1st observation: $g(a) \neq g(b)$. otherwise, by the Rolle's Theorem $\exists\, c \in (a,b)$ such that $g'(c) = 0$.

And this is not allowed because g is on the denominator

So $g(a) \neq g(b)$.

Now, let's construct the following continuous (on $[a,b]$) & differentiable (on (a,b)) function: h.

$$\boxed{h(x) = (f(b)-f(a))(g(x)-g(a)) - (g(b)-g(a))(f(x)-f(a))} \quad (*)$$

$$h(a) = (f(b)-f(a))\underbrace{(g(a)-g(a))}_{=0} - (g(b)-g(a))\underbrace{(f(a)-f(a))}_{=0} = 0$$

$$h(b) = (\underbrace{f(b)-f(a)}_{A})(\underbrace{g(b)-g(a)}_{B}) - (\underbrace{g(b)-g(a)}_{B})(\underbrace{f(b)-f(a)}_{A}) = 0$$

So $h(a) = h(b)$ & h continuous on $[a,b]$ & diff on (a,b)

By the Rolle's theorem, $\exists\, c \in (a,b)$ such that $h'(c) = 0$

$$h(x) = (f(b)-f(a))(g(x)-g(a)) - (g(b)-g(a))(f(x)-f(a))$$

$$h'(x) = (f(b)-f(a))\, g'(x) - (g(b)-g(a))\, f'(x)$$

$$h'(c) = (f(b)-f(a))\, g'(c) - (g(b)-g(a))\, f'(c) = 0$$

$$(f(b)-f(a))\cdot g'(c) = (g(b)-g(a))\, f'(c) \Longrightarrow$$

$$\frac{(f(b)-f(a))}{(g(b)-g(a))} = \frac{f'(c)}{g'(c)}$$

(*) Application:

Suppose f, g continuous on $[a,b]$.

If $\left[f'(x) = g'(x) \quad \forall x \in (a,b) \right]$ Then $\left[\exists C \in \mathbb{R} \text{ so that } g(x) = f(x) + C, \forall x \in I \right]$

<u>Proof</u>: Let f & g be two continuous functions on $[a,b]$ & differentiable function on (a,b)

Let's define $\boxed{h(x) = f(x) - g(x)}$
then h is also continuous on $[a,b]$ & differentiable on (a,b)

Thus $h'(x) = f'(x) - g'(x) = 0, \forall x \in (a,b)$
since by hypothesis $f'(x) = g'(x)$. $\forall x \in (a,b)$

Let $x_1, x_2 \in (a,b)$ with $x_1 < x_2$ so $[x_1, x_2] \subset (a,b) \subset [a,b]$.

By the Mean-Value Theorem $\exists\, c \in (a,b)$ such that

$$\frac{h(x_2) - h(x_1)}{x_2 - x_1} = h'(c) \equiv 0, \quad \forall c \in (a,b) \Longrightarrow$$

$$h(x_2) - h(x_1) = 0(x_2 - x_1) \Rightarrow h(x_2) = h(x_1) \quad \forall x_1, x_2 \text{ on } (a,b)$$

& by continuity $h(x) \equiv \text{constant } \forall x \in [a,b]$.
Let's call this constant C.

So $h(x) = f(x) - g(x) = C$
And therefore $f(x) = g(x) + C, \forall x \in [a,b]$ ∎

$h'(x) \equiv 0, \forall x \in (a,b)$
$\Downarrow$
$h(x) = C \in \mathbb{R}$

4. $\lim \quad (\sin x)$ ...

4. $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.}$

This does not work!

Obs: $|\sin x| < 1 \Rightarrow -1 < \sin x < 1 \Rightarrow$

$$-\frac{1}{x} < \frac{\sin x}{x} < \frac{1}{x}$$ TRUE But... unless...

THIS DOES NOT WORK!

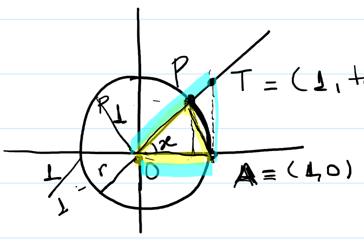
But this is not usefull since we want $x \to 0$ and in this case

$$\lim_{x\to 0} \frac{1}{x} = \begin{cases} +\infty & (\text{if } x \to 0_+ \\ -\infty & (\text{if } x \to 0_- \end{cases}$$

$$\lim_{x\to 0} -\frac{1}{x} = \begin{cases} -\infty & \text{if } x \to 0_+ \\ +\infty & \text{if } x \to 0_- \end{cases}$$

So it is impossible to use the Squeeze Theorem.

Let's recall Theo 8 page 122, where the correct proof is presented.



P= (cos x, sin x)

Proof $0 < x < \pi/2$.

A & P are points on $x^2+y^2=1$.

The area of the circular sector $\widehat{OAP}$ lies between the areas of ΔOAP & ΔOAT

Area ΔOAP < Area of sector < Area ΔOAT

$$\frac{1}{2}(1).\underbrace{\sin x}_{h} < \frac{x}{2}.(1) < \frac{1}{2}(1)\tan x = \frac{\sin x}{2\cos x}$$

$$\boxed{1 < \frac{x}{\sin x} < \frac{1}{\cos x}} \Longrightarrow$$

$$\boxed{1 > \frac{\sin x}{x} > \cos x}$$

$$1 = \lim_{x\to 0} 1 > \lim_{x\to 0} \frac{\sin x}{x} > \lim_{x\to 0} \cos x = 1.$$

Now by the Squeeze Theorem we have

$$\lim_{x\to 0} 1 = 1 > \lim_{x\to 0} \frac{\sin x}{x} > \lim_{x\to 0} \cos x = 1$$

$$\lim_{x\to 0} 1 = 1 \;\geq\; \lim_{x\to 0}\frac{\sin x}{x} \;\geq\; \lim_{x\to 0}\cos x = 1$$

$= 1$ $\qquad$ $= 1$

Área $= \frac{1}{2}\cdot b\cdot h$

Área $= b \times h$

Thus $\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1}$

final observation: (important to guarantee that $\left.\begin{matrix}\lim_{x\to 0_+}\frac{\sin x}{x} \\ \& \\ \lim_{x\to 0_-}\frac{\sin x}{x}\end{matrix}\right] =$

$\left.\begin{matrix} \sin x = -\sin(-x) \\ x = -(-x)\end{matrix}\right\}$ odd functions

Therefore $f(x) = \frac{\sin x}{x}$ is an even function $f(-x) = f(x)$

This symmetry implies that the left limit at 0 has the same value as the right limit

So $\lim_{x\to 0_-}\frac{\sin x}{x} = \lim_{x\to 0_+}\frac{\sin x}{x} = 1$ ∎

Taylor polynomials P_n

1) Linearization: (P_1 = Taylor Polynomial of degree 1)

Theo 11 page 272 (An error formula for linearization)

If $\left[\; \exists\, f''(t),\ \forall t\in I,\ a, x\in I \;\right]$

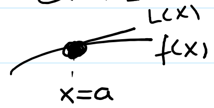
Then $\left[\begin{array}{l} \exists\, s \quad a < s < x \text{ such that the error} \\ E(x) = f(x) - L(x) \text{ in the linear approximation } f(x)\approx L(x) = f(a) + f'(a)(x-a) \\ \text{satisfies} \\ \qquad E(x) = \dfrac{f''(s)}{2!}(x-a)^2 \end{array}\right.$

Proof: (This is an application of the Generalized Mean-Value Theorem)

Assume $x > a$ (analogous for $x < a$)

$E(t) = f(t) - f(a) - f'(a)(t-a)$ $\qquad$ $(E(t) = f(t) - L(t))$ $\qquad$ $E(a) = 0$

$E(a) = f(a) - \cancel{f(a)} - \cancel{f'(a)(a-a)}$



(*) $E'(t) = f'(t) - L'(t) = \boxed{f'(t) - f'(a)}$

This is the function we are going to,

$\quad E(t) \qquad E(\ldots)$

$(*)\ E(t) = f(t) - L(t) = f(t) - f(a)$

Now assume $F(t) = E(t)$, $G(t) = (t-a)^2$

This is the function we are going to consider for the Generalized M-V-Theo.

$$\frac{F(t)}{G(t)} = \frac{E(t)}{(t-a)^2}$$

$$\frac{E(x)}{(x-a)^2} = \frac{E(x) - 0}{(x-a)^2 - (a-a)^2} = \frac{E(x) - E(a)}{(x-a)^2 - (a-a)^2} \overset{\text{By the generalized MV Theorem}}{=} \exists\, u \in (a,x)\ /$$

$$((x-a)^2)' = 2(x-a)\big|_{x=u}$$

$$= \frac{E'(u)}{\underbrace{2(u-a)}_{G'(u)}} \overset{(*)}{=} \frac{f'(u) - f'(a)}{2(u-a)} = \frac{1}{2}\left[\frac{f'(u) - f'(a)}{(u-a)}\right] \overset{(*)}{=} \frac{1}{2} f''(s)$$

By the Mean Value Theorem

Thus $\dfrac{E(x)}{(x-a)^2} = \dfrac{1}{2} f''(s) \Rightarrow \boxed{E(x) = \frac{1}{2} f''(s)(x-a)^2}$ ☒

$s \in (a,x)$

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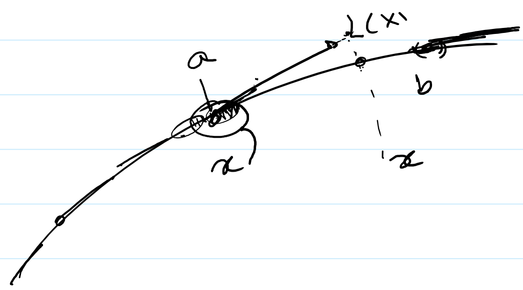
Corollary A $(*)$

If $\left[\begin{array}{l} f''(t) > 0;\ \forall t \in (a,x) \\ \hline f''(t) < 0;\ \forall t \in (a,x) \end{array}\right]$ Then $\left[\begin{array}{l} f(x) > L(x) \\ \hline f(x) < L(x) \end{array}\right]$ then $\left[\begin{array}{l} L(x) + E_1(x) > f(x) > L(x) \\ \hline L(x) - E_1(x) < f(x) < L(x) \end{array}\right]$

Corollary B

If $\left[\,|f''(t)| < K \quad \forall t \in (a,x),\ K \in \mathbb{R}_+\right]$

then $\left[\,|E(x)| < \frac{1}{2} K \cdot (x-a)^2\right]$



Example 5 page 273 (Important)

$f(x) = \cos(x)$
$\boxed{x = 36°}$

1) Use a linearization to find an approximation to $\cos(36°) = \cos(\pi/5)$

2) $\cos(36°) > L(36°)$ or $\cos(36°) < L(36°)$?

3) Estimate the size of the Error.

$\boxed{30° = a}$

Solution:

Consider $a = 30° = \pi/6$ (since it is near $x = 36° = \pi/5$)

Consider $a = 30° = \pi/6$ (since it is near $x = \underline{36°} = \pi/5$)

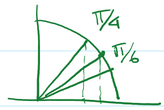
$$f(x) = \cos x$$

$$\begin{cases} f(x) \approx L(x) = f(\pi/6) + f'(\pi/6)(x - \pi/6) \\ E_1(x) = \dfrac{f''(s)}{2!}(x-\pi/6)^2 \end{cases}$$

$f(x) = \cos x$

$f'(x) = -\sin x$

$\boxed{f''(x) = -\cos x}$ $\qquad |f''(x)| < k \quad \forall x \in (\pi/6, \pi/4)$



$f'''(x) = \sin x$ $\qquad |f''(x)| < f''(\pi/6) = \frac{\sqrt{3}}{2}$ $(\cos 30°)$

So.

i) $\underset{\cos(36°)}{\overset{\shortparallel}{f(\pi/5)}} \cong f(\pi/6) + \dfrac{f'(\pi/6)}{1!}(\pi/5 - \pi/6)$

And $E_1(\pi/5) = \left| \dfrac{f''(s)}{2!}(\pi/5 - \pi/6)^2 \right| < \dfrac{\sqrt{3}}{2} \cdot \dfrac{1}{2}\left(\dfrac{\pi}{5} - \dfrac{\pi}{6}\right)^2$

$\boxed{E_1(\pi/5)} = \dfrac{\sqrt{3}}{4}\left(\pi/30\right)^2 = \boxed{\dfrac{\sqrt{3}\pi^2}{3600}}$.

$f''(x) < 0 \Rightarrow f(x) < L(x) \Longrightarrow \boxed{L(x) - E_1(x) < f(x) < L(x)}$

so $\cos(36°) \approx \cos(30°) + \dfrac{(-\sin(30°))}{1!}(36° - 30°)$

$$\approx \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{\pi}{5} - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\frac{\pi}{30} =$$

$$\frac{\sqrt{3}}{2} - \pi/60 = 0{,}8136\underline{65526225}\ldots \approx 0{,}8136\underline{7}$$

$$f(x) \in \left(0{,}81367 - \frac{\sqrt{3}\pi^2}{3600},\ 0{,}81367\right)$$

Theo 12 page 278 (Another Proof that considers the Generalized Mean-Value Theo)

Taylor's Theorem

If $\begin{cases} f, f', \ldots, f^{(n+1)}(t) \text{ exist } \forall t \in (a, x) \\ P_n(x) = \text{Taylor polynomial of degree } n \text{ for } f \text{ around } a \\ P_n(x) = f(a) + \dfrac{f'(a)}{1!}(x-a)^1 + \dfrac{f''(a)}{2!}(x-a)^2 + \ldots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n \end{cases}$

$P_n(x)$ = Taylor polynomial

$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Then the error $E_n(x) = f(x) - P_n(x)$ is given by

$$E_n(x) = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}}_{\text{Lagrange remainder.}} \qquad \xi \in (a, x)$$

page 280 Def: Big-O Notation

$f(x) = O(u(x))$ as $x \to a$ if $|f(x)| \le k|u(x)|$ $x \in I$, $a \in I$.

$f(x) = g(x) + O(u(x))$ as $x \to a$ if $|f(x) - g(x)| < k|u(x)|$ near a.

Theorem 13

If $f(x) = Q_n(x) + O((x-a)^{n+1})$ as $x \to a$, Q_n = pol of degree $\le n$

Then $Q_n = P_n$ = Taylor Polynomial

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{(n+1)} \le K(x-a)^{n+1} = O((x-a)^{n+1})$$