MSA101/MVE187 2020 Lecture 1.1
From Data to Decisions

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From data to decisions

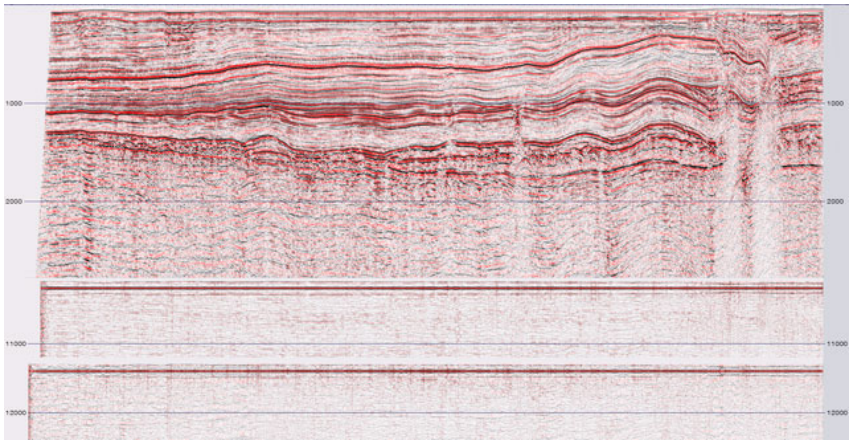
- Data
- Models
- Predictions
- Decisions

Various approaches for going from data to decision

- Non-probabilistic approach
  - Models describe generalizations of earlier experience.
  - Can we be certain the same will happen again?
  - What about observed variability in outcome?
- Probabilistic approach: Classical (frequentist) statistics
  - Select a set of *probabilistic* or *stochastic* models using context knowledge.
  - Estimate parameters and choose models using, e.g., *significance testing*.
  - Make predictions from models and select decisions based on predictions.
  - Some ad-hoc choices need to be done.
- Probabilistic approach: Bayesian statistics
  - Using probabilistic models for *knowledge* about something, not for the thing itself.
  - Provides seamless general theory from data to decision, as we shall see.
  - Can be computationally challenging.

Conceptual example: North sea oil production

- Context: The data about some new oil field consists of *seismic data* and observations from a *test well* producing some oil.
- Questions: How much oil does the field contain? Exactly where is it located?
- Decisions: Where to drill production wells? How many? How big infrastructure?



A non-probabilistic approach

- Using experience and scientific knowledge, make a best-guess model of what the reservoir looks like.
- Can we really trust the predictions from such a model?
- Observations are very indirect, there are large uncertainties. And uncertainties in various parts of the model will interact!
- Generally one uses probability theory, and probabilistic models, to understand and model the uncertainties and their interactions.
- NOTE: There exists only one oil reservoir! Classical probability theory deals with *repeatable events*, but the reservoir is not "repeatable"??

A probabilistic classical statistics approach

- Based on general knowledge of reservoirs, make a probabilistic model, with unknown parameters.
- Use statistics to *estimate* the parameters from observed data.
- Make *probabilistic predictions* from the model. Use these to select optimal decisions.
- How to estimate the parameters? How good are the decisions? Can they be trusted?

A probabilistic Bayesian approach

- A *stochastic model* (probabilistic model) of "all" possible "reasonable" reservoirs is built.
- A *posterior stochastic model* which takes into account the observed data is obtained, simply by finding the conditional probability distribution.
- Predictions for the consequences of various decisions are obtained from the posterior model. Predictions automatically come with uncertainties.
- The decision with the *optimal expected benefits* can be computed (in principle).
- NOTE: As soon as the initial stochastic model has been built, there are no more methodological choices that need to be made.
- NOTE: *Knowledge* about the reservoir is modelled, not the reservoir itself!
- NOTE: Recommended decisions are necessarily optimal, as long as the original model of the original knowledge of the reservoir is correct. Trust boils down to evaluating whether this initial model is reasonable (and all computations are correct mathematically).

MSA101/MVE187 2020 Lecture 1.2 Bayesian vs. Frequentist: Some issues with classical statistics.

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Frequentist issue 1: Interpretation

Example:

- We assume the numbers 4.2, 5.6 and 4.6 is a random sample from a normal distribution with expectation μ and fixed variance 1.
  As the numbers have mean 4.8, a 95% confidence interval for μ can then be computed as

$$\left[4.8 - 1.96 \cdot \frac{1}{\sqrt{3}}, 4.8 + 1.96 \cdot \frac{1}{\sqrt{3}}\right] = [3.67, 5.93]$$

- A possible interpretation: If three numbers are resampled from the distribution many times, the re-computed confidence intervals will contain μ with probability 95%.
- Another common interpretation: The interval [3.67, 5.93] contains μ with 95% probability.

Some attitudes towards misinterpretations of the confidence interval

- People need to be better educated about the correct interpretation.
- I don't care: As long as I as a mathematician/scientist compute and present correct results, it is not my problem how it is interpreted.
- The difference between the two interpretations above is so small it is unimportant.
- Other?

Frequentist issue 2: Objectivity

Example:

- Assume we have a sequence of independent trials each resulting in success (1) or failure (0), with a probability of succes equal to p. Assume we have observed the following data:

$$0, 1, 0, 0, 1, 0, 0, 1$$

  We then make the estimate $3/8 = 0.375$ for p. How "good" is this estimate?
- It is often said that an *estimator* that is unbiased is "good". Is this estimator unbiased? It depends on which estimator we have used!
- Alternative 1: The estimator is: Make 8 trials, let X be the number of successes, and compute $\hat{p} = X/8$.
- Alternative 2: The estimator is: Make trials until you have produced 3 successful trials, let X be the number of trials you needed to do, and compute $\hat{p} = 3/X$.

Continuation of example

- Exercise: Prove that the estimator in alternative 1 is unbiased (easy), and that the estimator in alternative 2 is biased (more difficult).
- Our point here: If we use the biasedness of the *estimator* to judge whether the *estimate* 0.375 is good, the result depends on which estimator we are using, which depends on what went on in the head (the plans) of the person doing the experiments.

Continuation of example

- In the same situation as above, and the same observations, we want to make a hypothesis test with $H_0 : p \geq 0.6$, and alternative hypothesis $H_1 : p < 0.6$. What is the p-value?
- To answer the question, we need to know which *test statistic* should be used.
- Alternative 1: The test statistic is: Make 8 trials and let X be the number of successes. Then, assuming $p = 0.6$, we get $X \sim \text{Binomial}(8, 0.6)$. The possible values for X and their probabilities are

0	1	2	3	4	5	6	7	8
0.001	0.008	0.041	0.124	0.232	0.279	0.209	0.090	0.017

We get that the p-value becomes 0.174; the sum of the probabilities for $X = 0, 1, 2, 3$.

Continuation of example

- Alternative 2: The test statistic is: Make trials until 3 successes have appeared and let X the number of trials necessary. Then, assuming $p = 0.6$, we get $X \sim$ Neg-Binomial$(3, 0.6)$. The possible values for X and their probabilities are

3	4	5	6	7	8	9	10	11
0.216	0.259	0.207	0.138	0.083	0.046	0.025	0.013	0.006

12	13	14	15	16,17,. . .
0.003	0.001	0.001	0.000	total 0.000

We get the p-value 0.095; the sum of the probabilities for $8, 9, 10, \ldots$.

- Note that if we use a significance level of 0.1, we will reject the null hypothesis using the second test statistic, but not using the first test statistic.

MSA101/MVE187 2020 Lecture 1.3 Bayesian vs. Frequentist: Some issues with classical statistics.

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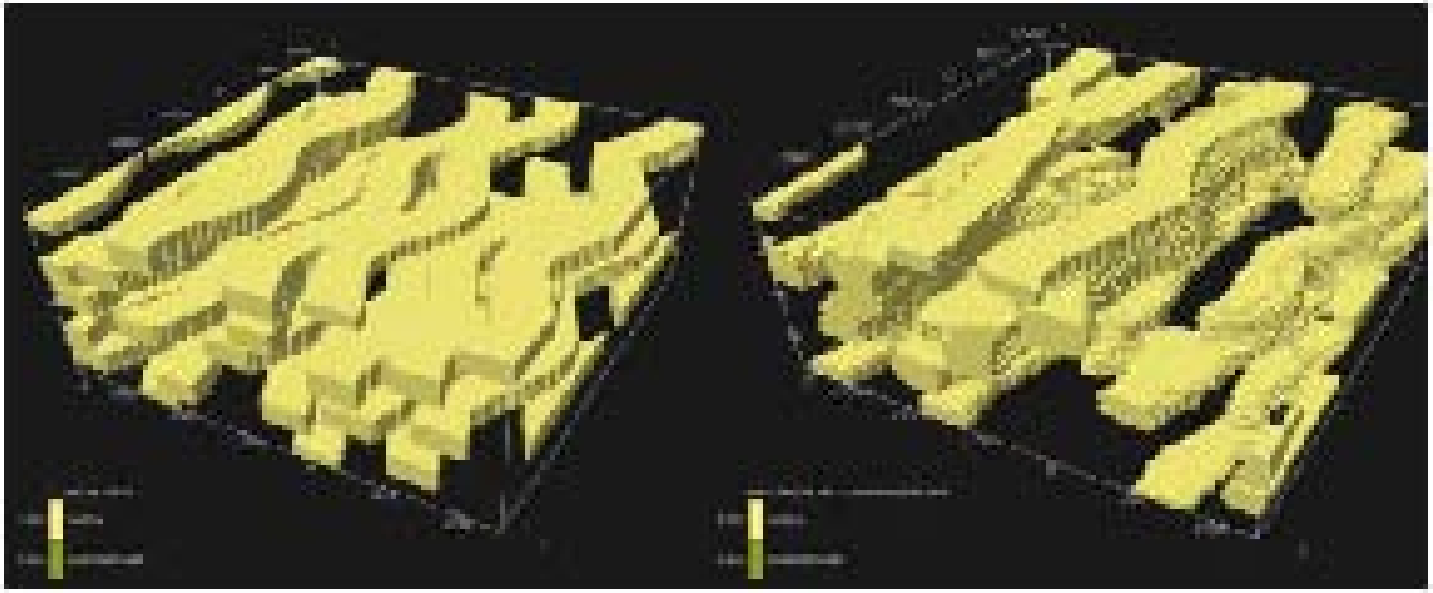
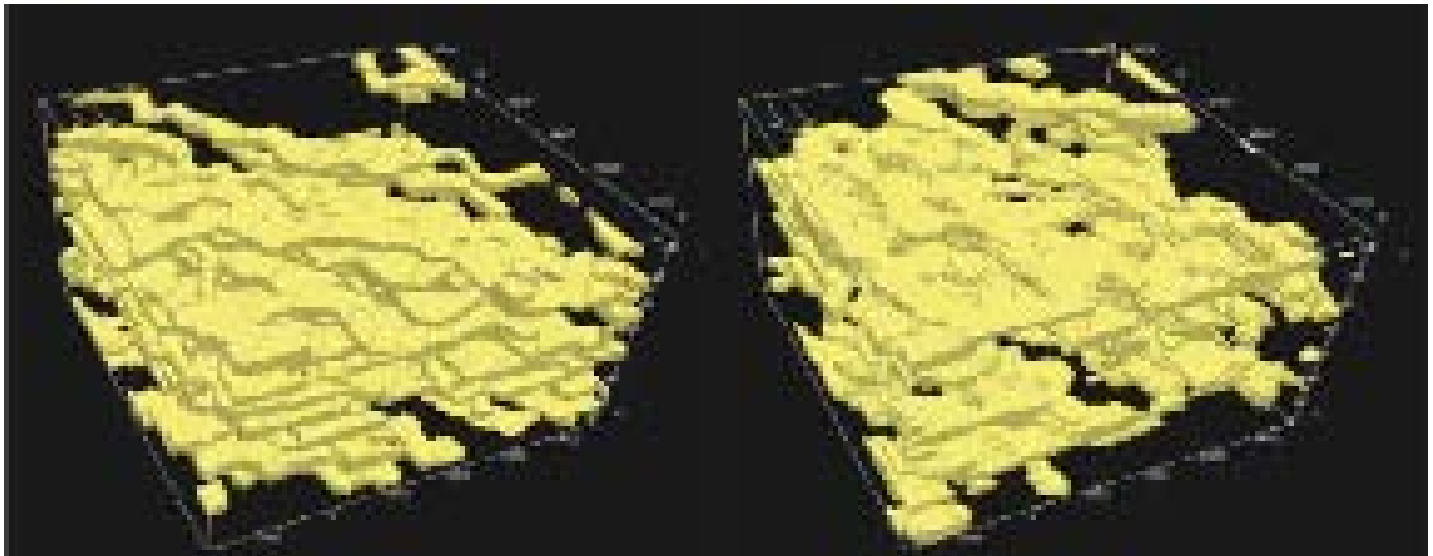
Frequentist issue 3: Contextual information

- Assume you want to find out if a coin is "fair", i.e., if the probability p for heads is 0.5. You throw the coin 8 times and get heads 2 times. What do you believe about the probability p, and how certain can you be?
- Assume you are a doctor who has received permission for a new experimental surgical procedure. After 8 procedures, 2 are unsuccessful. What do you believe about the probability p for an unsuccessful procedure, and how certain can you be?
- Assume you work at a factory and you want to make a quality control of a product. Out of 8 randomly chosen items, 2 were faulty. What do you believe about the probability p that an item is faulty, and how certain can you be?
- We saw earlier that what people generally want from a statistical analysis are probabilistic predictions about future observations. Generally, such predictions will need to take the context into account. If p is simply regarded as an "unknown parameter", this cannot be done.

Frequentist issue 4: Repeatability

Example: Stochastic modelling of oil reservoirs: There is only ONE reservoir, how can one talk about probabilities?





The Bayesian paradigm for statistics

- A set of variables (discrete and/or continuous) are chosen to represent or describe some part of the real world, The variables include
  - Variables representing observed quantities.
  - Variables representing things you want to know or predict.
  - Ancillary variables.
- A function over all possible combinations of values of the variables is established, representing a joint probability distribution: This represents our knowledge *before* we have looked at the data.
- Some of the variables are observed (i.e., fixed) and the probability model conditional on fixing these variables is found.
- Predictions for observable quantities are made from the conditional model.

Some comments on the Bayesian approach

- Our goal is to build *stochastic models* (probabilistic models) for the real world, corresponding to our knowledge, and to use these models to make probabilistic *predictions*.
- Probability is a feature of *knowledge* of the real world, not of the real world itself.
- It is not useful to try to separate between "unknown parameters" and "random variables" in these models: All are known/unknown to some extent, and they should all be treated as random variables.
- The stochastic models are *personal* (as they model knowledge), but rational persons with the same knowledge about some part of reality should obtain the sams stochastic models for that part of reality.

Bayesian comments on example from frequentist issue 3

- Note that *predictions* are key!
- Note that predictions might vary, in that they might use different information. They are personal. There is no "correct" prediction.
- You would model the probability p for a successful procedure as a random variable, not as a parameter.
- You would model the probability density for p *without* taking into account the experiment where 2 of 8 procedures succeded: The *prior* for p.
- You then compute the probaility density for p *taking into account* the dat from the experment: This is the *posterior* for p.

Statistics as learning, not "estimation"

- Assume a stochastic model includes a variable X modelling some real world quantity. Assume that quantity is observed to have the value x. Then our *updated model* should be the stochastic model *conditioned on* the information $X = x$.
- Technically, this conditioning will correspond to using Bayes theorem, which is why this is called Bayesian statistics.
- In fact, all scientific learning is based on making observations. If a scientific theory is represented as a stochastic model, the process of scientific learning can be represented, to a certain approximation, as a Bayesian update of this model.

Example: Stochastic modelling of oil reservoirs

- The variable of interest might be the amount of oil, the data might be geological observations along a trial well. Many other variables describe the geological geometry.
- Not useful to estimate "parameters" from data: Knowledge about geological geometry will only increase somewhat with this particular data.
- Important to take the residual uncertainty in parameters into consideration, when predicting!

Frequentist vs Bayesian statistics

- The frequentist and Bayesian paradigms, when used on the same problem, often yield similar or identical practical results. Why?
- The two methods should share the same *likelihood model*. A frequentist approach for estimation followed by prediction in many cases correspond computationally to a particular choice of prior distribution on the parameters. When this prior corresponds to the one used in the Bayesian analysis, the two approaches give similar results in practice.
- Example: Learning about a proportion p from repeated experiments. A uniform prior on $[0, 1]$ yields Bayesian results corresponding to classical ones.

Example: Intervals for expectations of normal distributions

- We assume data $x_1, \ldots, x_n$ is a random sample from a normal distribution with expectation μ and known variance $\sigma^2 = 1$.
- A frequentist analysis can compute from $x_1, \ldots, x_n$ a 95% confidence interval, say $[0.42, 0.73]$, for μ.
- People tend to interpret this as $P(0.42 \leq \mu \leq 0.73) = 0.95$. (This interpretation is not correct).
- However, if we assume a *flat prior* for μ and do a Bayesian analysis, we derive at the 95% *credibility interval* $[0.42, 0.73]$. The correct interpretation of this is exactly $P(0.42 \leq \mu \leq 0.73) = 0.95$.
- Note: We here expand the set of probability distributions to include also *improper distributions*, i.e., those that integrate (or sum) to ∞.
- For many, but not all, situations, a flat prior may be reasonable.