Foundations of Probability Theory
(MVE140 – MSA150)

Saturday 16th of January 2021 examination questions

This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points (50 if you are a PhD student). You should keep a zoom session with a camera on showing you working for the whole duration of the exam. A recording will be made which will be deleted soon after the results are released.

Examination Questions

1. An urn contains N balls numbered from 1 to N. A ball is drawn at random from the urn, its number is recorded. Let X be the maximal number after n draws. Find the distribution of X when

   a) the drawn ball is returned back to the urn after its number is recorded;

   b) the drawn ball is removed from the subsequent draws. In this case, it is assumed that $n \leq N$.

Solution.

a) Note that $X \leq k, k = 1, \dots, N$ when each time only the balls with numbers 1 to k were drown, so the probability of this is $(k/N)^n$. Then

$$\mathbf{P}\{X = k\} = \mathbf{P}\{X \leq k\} - \mathbf{P}\{X \leq k-1\} = \frac{k^n - (k-1)^n}{N^n}, \; k = 1, \dots, N.$$

b) For $X = k$, $n \leq k \leq N$ the ball numbered k must be drawn and the other $n-1$ balls must be within the numbers 1 to $k-1$. There are $\binom{k-1}{n-1}$ such choices. Overall, there are $\binom{N}{n}$ variants, all equiprobable, thus

$$\mathbf{P}\{X = k\} = \frac{\binom{k-1}{n-1}}{\binom{N}{n}}, \quad n \leq k \leq N.$$

2. Let ξ_i, ξ_2 be two independent Binomially distributed random variables with parameters (n_1, p) and (n_2, p) respectively (p is the *same* for both). Find

   a) the conditional distribution of ξ_1 given their sum $S = \xi_1 + \xi_2 = m,\ 0 \le m \le n_1 + n_2$;

   b) the conditional expectation $\mathbf{E}[\xi_1|S]$ (you might wish to consider the indicators χ_i of the success in the i-th trial).

Solution.

a) Since $S \sim \mathsf{Bin}(n_1 + n_2, p)$ then, using independence, for $0 \le k \le m$,

$$\mathbf{P}\{\xi_1 = k \bigm| S = m\} = \frac{\mathbf{P}\{\xi_1 = k, \xi_2 = m - k\}}{\mathbf{P}\{S = m\}}$$
$$= \frac{\binom{n_1}{k} p^k (1-p)^{n_1 - k} \binom{n_2}{m-k} p^{m-k} (1-p)^{n_2 - m + k}}{\binom{n_1+n_2}{m} p^m (1-p)^{n_1+n_2-m}} = \frac{\binom{n_1}{k}\binom{n_2}{m-k}}{\binom{n_1+n_2}{m}}$$

which is independent of p.

b) Let $\chi_i,\ i = 1, \ldots, n_1 + n_2$, be the indicators of the success in he i-th trial. Then

$$\mathbf{P}\{\chi_i = 1 \bigm| S = m\} = \frac{\binom{n_1+n_2}{m-1}}{\binom{n_1+n_2}{m}} = \frac{m}{n_1 + n_2}.$$

For this, you either repeat the above reasoning or, since χ_i's are identically conditionally distributed, just use the previous distribution with $k = 1,\ n_1 = 1$ and $n_2 = n_1 + n_2 - 1$. Thus $\mathbf{E}[\chi_i \bigm| S = m] = m/(n_1 + n_2)$, i.e. $\mathbf{E}[\chi_i \bigm| S] = S/(n_1 + n_2)$, implying

$$\mathbf{E}[\xi_1 \bigm| S] = \sum_{i=1}^{n_1} \mathbf{E}[\chi_i \bigm| S] = \frac{S n_1}{n_1 + n_2}$$

3. Let $\xi_1, \xi_2, \ldots$ is a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Show that the set $C = \{\omega \in \Omega : \ \xi_n(\omega) \text{ converges}\}$ is an $\mathcal{F}$-measurable set. Show that there exists a random variable ξ (i.e. an $\mathcal{F}$-measurable mapping from Ω to $\mathbb{R}$) such that $\xi(\omega) = \lim_{n\to\infty} \xi_n(\omega)$ for all $\omega \in C$.
Solution. By the Cauchy criterion, a sequence of numbers $\xi_n(\omega)$ converges if for all $k \in \mathbb{N}$ there is an n such that for all $m_1, m_2 \geq n$ one has $|\xi_{m_1}(\omega) - \xi_{m_2}(\omega)| < 1/k$. Thus

$$C = \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m_1\geq n} \bigcap_{m_2\geq n} \{\omega : \ |\xi_{m_1}(\omega) - \xi_{m_2}(\omega)| < 1/k\}.$$

The difference of measurable functions is a measurable function, therefore $\{\omega : \ |\xi_{m_1}(\omega) - \xi_{m_2}(\omega)| < 1/k\} \in \mathcal{F}$ and hence $C \in \mathcal{F}$ as a countable union and intersections of measurable sets.
Next, for all $\omega \in C$ then there exist a number $\xi(\omega)$ also generally depending on ω, such that $\xi(\omega) = \lim_{n\to\infty} \xi_n(\omega)$. Set, for instance, $\xi(\omega) = 0$ for $\omega \in \Omega \setminus C$. Then, for $x < 0$,

$$\begin{aligned} \xi^{-1}((-\infty, x]) = \{\omega : \ \xi(\omega) \leq x\} = C \bigcap \{\xi \leq x\} \\ = C \bigcap \cap_k \cup_n \cap_{m\geq n} \{\xi_m \, \mathbb{I}_C \leq x + 1/k\} \in \mathcal{F} \end{aligned}$$

because ξ_m are measurable functions for all m. Similarly, for $x \geq 0$,

$$\{\omega : \ \xi(\omega) \leq x\} = C \cap \{\xi \leq x\} \cup C^{\mathbf{c}} \in \mathcal{F}.$$

Since $(-\infty, x]$ are generating sets for the Borel σ-field, then $\xi^{-1}(B) \in \mathcal{F}$ for all Borel B, i.e. ξ is a random variable.

4. Let ξ and η be independent random variables each having Exponential $\mathrm{Exp}(\lambda)$ distribution. Denote $\zeta = \xi + \eta$. Find the joint density of the pair (ξ, ζ) and deduce that the conditional density of ξ given $\zeta = t$ corresponds to the uniform distribution on $(0, t)$. In other words, knowing $\xi + \eta$ bears no information on the value of ξ! Find $\mathbf{E}[\xi \bigm| \zeta]$ and the expectation of it.

Solution. The density of ξ (and also of η) is $f_\xi(x) = \lambda e^{-\lambda x} = f_\eta(x),\ x \geq 0$. The conditional density of $\xi + \eta$ given $\xi = x$ corresponds to the density of $x + \eta$ so it is $f_{\xi+\eta|\xi}(t|x) = \lambda e^{-\lambda(t-x)}$ for $t \geq x$ and 0 otherwise. Thus the joint density is

$$f_{\xi,\xi+\eta}(x,t) = f_{\xi+\eta|\xi}(t|x) f_\xi(x) = \lambda^2 e^{-\lambda t}\, 1\!\!\mathrm{I}\{0 \leq x \leq t\}.$$

Therefore,

$$\begin{aligned} f_{\xi|\xi+\eta}(x|t) &= f_{\xi,\xi+\eta}(x,t)/f_{\xi+\eta}(t) \\ &= f_{\xi,\xi+\eta}(x,t)\left[\int_0^t f_{\xi,\xi+\eta}(x,t)\,dx\right]^{-1} \\ &= \lambda^2 e^{-\lambda t}\left[t\lambda^2 e^{-\lambda t}\right]^{-1} 1\!\!\mathrm{I}\{0 \leq x \leq t\} = t^{-1}\, 1\!\!\mathrm{I}\{0 \leq x \leq t\}. \end{aligned}$$

The density (a function of x!) is a constant t^{-1} on the interval $[0,t]$, i.e. the distribution is uniform. Its mean is $t/2$ so that $\mathbf{E}[\xi \mid \zeta] = \zeta/2$. By the Full expectation formlula, $\mathbf{E}\,\mathbf{E}[\xi \mid \zeta] = \mathbf{E}\,\xi = 1/\lambda$. It is also clear from $\mathbf{E}\,\zeta/2 = (\mathbf{E}\,\xi + \mathbf{E}\,\eta)/2 = 1/\lambda$.

5. Let $\{\xi_n\}$ be a sequence of random variables with the following distribution symmetrical with respect to a point a: ξ_n takes values $-n^\alpha + a$ and $n^\alpha + a$ for some α with equal probabilities. Characterise the sequences of normalising constants $\{c_n\}$ for which the sequence $c_n\xi_n$ has a weak limit. When does this limit is non-trivial (i.e. it is not a constant)?
Solution. The characteristic function: $\varphi_{\xi_n}(t) = e^{iat}(0.5e^{-itn^\alpha} + 0.5e^{itn^\alpha}) = e^{iat}\cos(tn^\alpha)$. Thus $\varphi_{c_n\xi_n}(t) = e^{iac_nt}\cos(tc_nn^\alpha)$ which has a limit as $n \to \infty$ iff both terms have a limit, i.e. when $c_nn^\alpha \to C_1 < \infty$ and $c_n \to C_2 < \infty$. Thus, either $\alpha < 0$ and $0 \leq C_2 < \infty$ or $\alpha = 0$ and $C_2 = 0$ or $\alpha > 0$ and $C_2 = 0$, then $\varphi_{c_n\xi_n}(t) \to e^{iaC_2t}$, i.e. the limit is trivial corresponding to the constant aC_2. Alternatively, either $\alpha = 0$ and $0 < C_2 < \infty$, then $\varphi_{c_n\xi_n}(t) \to e^{iaC_2t}(0.5e^{-it} + 0.5e^{it})$, i.e. the limit is a random variable taking values $aC_2 - 1$ and $aC_2 + 1$ with equal probabilities. Or $\alpha > 0$ and $0 < C_1 < \infty$, in which case $\varphi_{c_n\xi_n}(t) \to (0.5e^{-itC_1} + 0.5e^{itC_1})$, i.e. the limit is a symmetric random variable taking values $-C_1$ and C_1 with equal probabilities.