Föreläsning_5.2_alice

den 3 december 2020 08:00

Mål:

- Underrum $\rightarrow$ Subspace
- Nollrum $\longrightarrow$ Nullspace (Null(A))
- Kolunnrum $\longrightarrow$ Columnspace (Col(A))
- Rank, Bas, dimension, koordinater $\longrightarrow$ Rank, Base, dimension, coordinates.

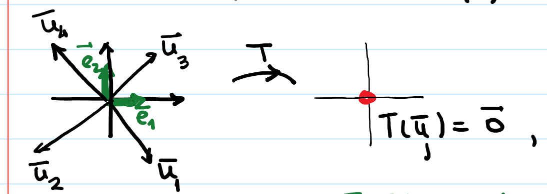
We will start with many examples of linear transforms $T: V \rightarrow W$, $\bar{u} \rightarrow T(\bar{u})$

Recall: T is linear if $\begin{cases} T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v}), & \forall \bar{u}, \bar{v} \in V = \text{domain space} \\ T(\lambda \bar{u}) = \lambda T(\bar{u}), & \forall \lambda \in \mathbb{R}, \ \forall \bar{u} \in V. \end{cases}$

Examples:

① $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$\vec{u} \rightarrow T(\bar{u}) = 0.\vec{u} = (0,0)$



$T(\bar{e}_1) = (0,0)$
$T(\bar{e}_2) = (0,0)$

$\bar{u} = (x, y) \in \mathbb{R}^2 = V$

$\bar{u} = (x,y) = (x,0) + (0,y) = x(1,0) + y(0,1)$

$\bar{u} = x\,\bar{e}_1 + y\,\bar{e}_2$

$$T(\bar{u}) = T(x\bar{e}_1 + x\bar{e}_2) = x\,T(\vec{e}_1) + y\,T(\vec{e}_2)$$

$$= x\begin{bmatrix} 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 0 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{A_{2\times2}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{[\bar{u}]_{2\times1}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{[T(\bar{u})]_{2\times1}}$$

a) T 1:1 ? No: $\bar{e}_1 \neq \bar{e}_2$ but $T(\bar{e}_1) = T(\bar{e}_2) = (0,0)$

b) T onto-mapping? No, $\exists\ \bar{w} = (1,1) \in W = \mathbb{R}^2$ but $T(\bar{u}) \neq \bar{w} = (1,1)\ \forall \vec{u} \in \mathbb{R}^2 = V$

(No vector $\vec{u}$ from the domain goes to $\bar{w} = (1,1)$ in the Codomain)

c) T bijection? No : T is NOT 1:1 NOR ONTO.

(d) . What vectors $\vec{X} \in V$ = domain that satisfy $\underline{A X = \vec{0}}$?
. What are the solutions of the homogeneous system $AX = \bar{0}$?
. What vectors from V are brought to $\vec{0}$ by the transform T?

$$AX = 0 \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 0x + 0y = 0 \\ 0x + 0y = 0 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ 0 = 0 \end{cases}$$

So there is no equation involving $x \Rightarrow x$ free
same to $y \Rightarrow y$ is also a free parameter:

So $X = \begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

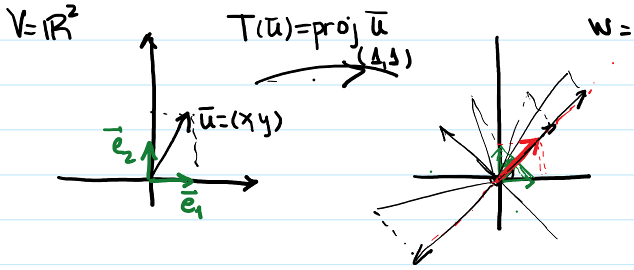
$\alpha, \beta \in \mathbb{R}$

So the solution of the Homogeneous system $AX = 0$ is the set of all linear combinations of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Solution of $AX = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

2. $T : \mathbb{R}^2 \to \mathbb{R}^2$

$\bar{u} \to T(\bar{u})$ = projection of $\bar{u}$ in the direction of $(1,1)$ $= \text{proj}_{(1,1)} \bar{u}$



$\bar{u} = (x\,y)$

$T(\bar{u}) = \text{proj}_{(1,1)}(\bar{u}) = \dfrac{(x,y)\cdot(1,1)}{(1,1)\cdot(1,1)}(1,1) =$

$= \dfrac{x+y}{2}(1,1) =$

$T(\bar{u}) = \left(\dfrac{x}{2} + \dfrac{y}{2}, \dfrac{x}{2} + \dfrac{y}{2}\right)$

$T(\bar{e}_1) = T(1,0) = \left(\frac{1}{2} + 0, 0 + \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$

$T(\vec{e}_2) = T(0,1) = \left(0 + \frac{1}{2}, 0 + \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$

$\bar{u} = (x,y) = x\bar{e}_1 + y\bar{e}_2$

$T(\bar{u}) = T(x\,y) = xT(\bar{e}_1) + yT(\bar{e}_2) = x\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + y\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

Matrix form of T

$$\underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}_{A_{2\times 2}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{[\bar{u}]_{2\times 1}} = \underbrace{\begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x + \frac{1}{2}y \end{bmatrix}}_{[T(\bar{u})]_{2\times 1}}$$

a) T 1:1 ? No! $\bar{e}_1 = (1,0) \neq \bar{e}_2 = (0,1)$ but $T(\bar{e}_1) = (1/2, 1/2) = T(\bar{e}_2)$

b) T ONTO mapping? No! There exists $\bar{w} = (7,3) \in W = \mathbb{R}^2$ BUT NO vector $\vec{u} \in V = \mathbb{R}^2$ goes to $\bar{w}$ with the transform T.

c) So T is not a bijection (Because T is NOT 1:1, nor ONTO) and has no inverse. (Bijection $\Rightarrow$ invertible)

d) And Now what are the vectors brought to $\vec{0} \in W$ by T?
What are the solutions of the homogeneous system $AX = \vec{0}$?
? $\bar{u} \in V$ such that $T(\bar{u}) = \vec{0}$?

$T(x,y)=0 \iff \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Computing the solution by the Echelon form $\left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right] \; L_2 \leftarrow L_2 - L_1 \Rightarrow$

$\left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \; L_1 \leftarrow 2L_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$

the system solution is $\begin{array}{cc} \text{LD} & \\ \end{array} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 1x + 1y = 0 \\ 0 + 0 = 0 \end{cases} \Rightarrow$

$\Rightarrow x = -y$
and no equation for y $\therefore$ $y \equiv$ free parameter $= \lambda \in \mathbb{R}$

The general form of the solution is:

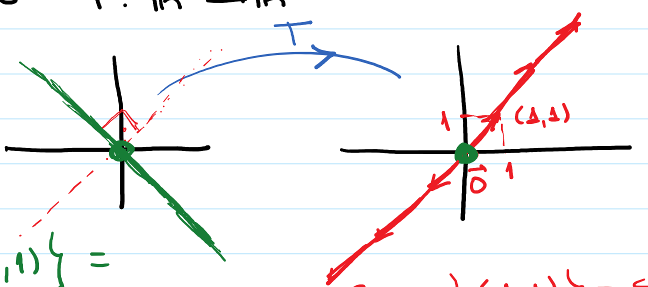
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \; \lambda \in \mathbb{R}.$$

So the set of all vectors X such that $AX = \vec{0}$ is

$$S = \{ (x,y) = \lambda(-1,1), \; \lambda \in \mathbb{R} \} = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} = \operatorname{Span}\{(-1,1)\}$$

obs: Here we are just playing with the Notation: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}_{2\times 1}$ matrix $\leftrightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix} \leftrightarrow (-1,1)$ vector

So $T: \mathbb{R}^2 \to \mathbb{R}^2$



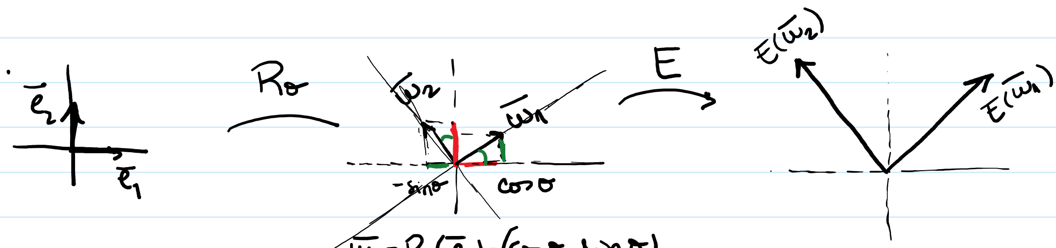
$\operatorname{Span}\{(-1,1)\} =$
$\operatorname{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} =$
Set of all solutions of $AX = \vec{0}$

$\operatorname{Span}\{(1,1)\} = \operatorname{Span}\{(1/2, 1/2)\} = \operatorname{Span}\left\{\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}\right\}$
$= \operatorname{Span}\{T(\vec{e}_1), T(\vec{e}_2)\}$

③ $T: \mathbb{R}^2 \to \mathbb{R}^2$ composed by 1st Rotation of $\theta = 30^\circ = \pi/6$ counter clockwise
followed by expansion in all directions

of a factor of 2.

$$T(\bar{u}) = E(R_\theta(u))$$



$$\bar{w}_1 = R_\theta(\bar{e}_1) = (\cos\theta, \sin\theta)$$
$$= (\cos\pi/6, \sin\pi/6)$$
$$\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right)$$

$$[E]_{2\times2} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\bar{w}_2 = \left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)$$

$$[R_\theta]_{2\times2} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$T(\bar{u}) = E(R_\theta(\bar{u})) = [E]\left([R_\theta][\vec{u}]\right) =$$
$$= ([E]\cdot[R_\theta])[\vec{u}]$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$
$$= \underbrace{\begin{bmatrix} \sqrt{3} & -1 \\ +1 & \sqrt{3} \end{bmatrix}}_{A_{2\times2}} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{3}x - 1y \\ 1x + \sqrt{3}y \end{bmatrix}}_{[T(\bar{u})]_{2\times1}}$$

$$A_{2\times2} \cdot [u]_{2\times1} = [T(\bar{u})]_{2\times1}$$

1) T 1:1 ? $\Leftrightarrow$ $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has only the trivial solution $\vec{x} = \vec{0}$.
(unique solution and the trivial one)

By the Echelon form we have: B = Echelon of $[A|0]$

$$\left[\begin{array}{cc|c} \sqrt{3} & -1 & 0 \\ +1 & \sqrt{3} & 0 \end{array}\right] \quad L_2 \leftarrow L_2 - \frac{1}{\sqrt{3}} L_1$$

$$\left[\begin{array}{cc|c} \sqrt{3} & -1 & 0 \\ 1 - \frac{1}{\cancel{\sqrt{3}}}(\cancel{\sqrt{3}}) & \sqrt{3} - \frac{1}{\sqrt{3}}(-1) & 0 - 0 \end{array}\right] = \left[\begin{array}{cc|c} \sqrt{3} & -1 & 0 \\ 0 & \frac{4\sqrt{3}}{3} & 0 \end{array}\right] \quad L_2 \leftarrow \frac{3}{4\sqrt{3}} L_2$$

$$\sqrt{3} + \frac{1}{3}\sqrt{3} = \sqrt{3} + \frac{\sqrt{3}}{3} = \frac{4}{3}\sqrt{3}$$

$\sqrt{3}$ $\quad$ $4\sqrt{3}$

$$\sqrt{3}+\frac{1\sqrt{3}}{\sqrt{3}\sqrt{3}} = \sqrt{3}+\frac{\sqrt{3}}{3} = \frac{4}{3}\sqrt{3}$$

$$\left[\begin{array}{cc|c} \sqrt{3} & -1 & 0 \\ 0 & 1 & 0 \end{array}\right] \quad L_1 \leftarrow L_1+L_2 \quad \left[\begin{array}{cc|c} \sqrt{3}+0 & -1+1 & 0+0 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cc|c} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \quad L_1 \leftarrow \tfrac{1}{\sqrt{3}} L_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{cases} 1x+0=0 \Rightarrow x=0 \\ 0+1y=0 \Rightarrow y=0 \end{cases}$$

LI $\Rightarrow$ The original vectors (**) are also LI.

So the solution of $A\vec{x}=\vec{0}$ is only $\bar{x}=\begin{pmatrix}0\\0\end{pmatrix}$ unique & Trivial solution

So now $T(\bar{u})=\vec{0}$ only if $\vec{u}=0$.

2) T is onto? $\quad A\bar{x}=[\bar{w}]$ has solution for all $\bar{w}\in W=\mathbb{R}^2$

What do we already know?

$\rightarrow$ $AX=b$ $\quad X=X_H+X_p$ with X_H = solution of $AX=0$ (unique)

X_p = particular solution (unique)

$\rightarrow$ Theorem 4 page 53 $\begin{cases} A \text{ has pivot position in every Row} \Leftrightarrow \\ \text{Columns of } A \text{ span } \mathbb{R}^2 \Leftrightarrow \\ AX=b \text{ has solution for all } \bar{b}\in\mathbb{R}^2 \end{cases}$

There fore T is ONTO.

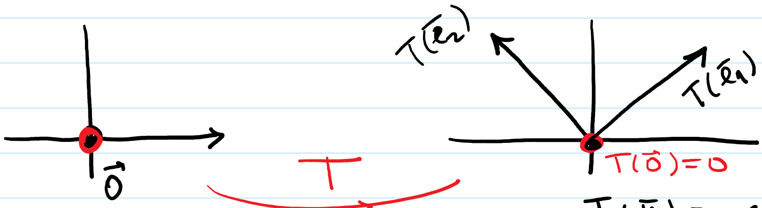
& more: Since B= Echelon form of $[A\,|\,0]$ is given by

$$B=\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow \text{Range}(T)=\text{Span}\left\{\begin{pmatrix}\sqrt{3}\\1\end{pmatrix},\begin{pmatrix}-1\\\sqrt{3}\end{pmatrix}\right\}$$

LI $\qquad$ The original columns of A.

Geometrically: all vectors of $\mathbb{R}^2=V$ are rotated by an angle of $\theta=30^\circ=\pi/6$ counterclockwise and after that are expanded by a factor of 2 in both directions $\overline{Ox}$ & $\overline{Oy}$.

Only $T(\bar{u})=\bar{0}$ only when $\bar{u}=\bar{0}\in V$.



$T(\bar{u})=xT(\bar{e}_1)+yT(\bar{e}_2)$

$$T(\bar{u}) = x\,T(\bar{e}_1) + y\,T(\bar{e}_2)$$

$$\text{Span}\{X\} = \{\vec{0}\}$$

X = solution of $AX = \vec{0}$

$$\text{Span}\{T(e_1), T(\bar{e}_2)\} = W = \mathbb{R}^2$$

So Now let's Define the objects we considered on these 3 examples:

Def: Given a Vector Space $V = \mathbb{R}^n$, with operations $+$, $\lambda\cdot$

A subset $S \subset V$ is called a subspace of V if

(i) $\bar{u}, \bar{v} \in S \Rightarrow \bar{u} + \bar{v} \in S$

(ii) $\lambda \in \mathbb{R}, \bar{u} \in S \Rightarrow (\lambda \bar{u}) \in S$

So: • Elements of S when summed or multiplied by a constant remain in S.

- Linear combinations of elements of S are still elements of S.
- S is "closed" for the operations of $(+)$ and $(\lambda\cdot)$

i) $T: V \longrightarrow W$

$\bar{u} \longrightarrow T(\bar{u}) = A\vec{u}$

$S = \text{Span}\{\bar{X}\}$, $\bar{X}$ solution of $A\bar{X} = \vec{0}$

S is a subspace of $V \equiv$ domain of T

S is called Nullspace(A) or Kernel of $T \equiv \ker(T)$

To prove this let's consider $X, Y \in S \therefore AX = 0$ and $AY = 0$

(i) ? $(X+Y) \in S$?

$$\underbrace{A(X+Y) = AX + AY = \vec{0} + \vec{0} = \vec{0}}$$

So, yes $(X+Y) \in S$
$(X+Y)$ is still a solution of the Homogeneous system $AZ = \vec{0}$

(ii) ? $\lambda X \in S$, if $X \in S$?

$$\underbrace{A(\lambda X) = \lambda(AX) = \lambda(\bar{0}) = \vec{0}}$$

Yes $(\lambda \bar{X})$ is another solution of the homogeneous system $AZ = \vec{0}$.

Def 2: $T: \mathbb{R}^n \to \mathbb{R}^m$ $\quad A_{m\times n} = \left[a_1 \ldots a_n\right]_{n\times m}$ $\quad a_i$ = column of A

$u \to T(\bar{u}) = A_{m\times n}\cdot [u]_{n\times 1}$

$Col(A) \equiv$ the column space of $A \equiv Span\{a_1, a_2, \ldots a_n\}$

To prove this: $T: \mathbb{R}^n \to \mathbb{R}^m$ $\quad T(\bar{u}) = A_{m\times n}[\bar{u}] = [a_1 \cdots a_n][u]$

$\vec{u} \to T(\bar{u})$

Assume $\vec{X}, \vec{Y} \in S = Col(A) = Span\{\vec{a_1}, \ldots \vec{a_n}\}$

i) So $\bar{x} = \alpha_1 a_1 + \cdots + \alpha_n a_n$ $\quad$ they are linear combinations of

$y = \beta_1 a_1 + \cdots + \alpha_n a_n$ $\quad \{\vec{a}_1 \cdots a_n\}$

$(x+y) = (\alpha_1+\beta_1)a_1 + \cdots + (\alpha_n+\beta_n)a_n$ $\quad \therefore$ again linear comb. $\{a_1 \ldots a_n\}$

ii) & $\lambda\bar{X} = \lambda(\alpha_1 a_1 + \cdots + \alpha_n a_n)) =$

$= (\lambda\alpha_1)a + \cdots + (\lambda\alpha_n)a_n$ $\quad \therefore$ again linear comb. of $\{a_1, \ldots, a_n\}$

$Col(A) \subset W$ subspace of W

Def 3 $\quad B = \{\bar{u}_1, \bar{u}_2, \ldots \bar{u}_n\}$ $\quad B \subseteq V$ = vector space or subvector space.

If $\begin{cases} B \text{ is } LI & \& \\ Span\{B\} = span\{\bar{u}_1, \ldots, \bar{u}_n\} = V \end{cases}$

Then B is called a Basis of V.

Def 4: V = vector space, B basis of V.

Dimension of $V \equiv$ nº of vectors of B.

So $V = \mathbb{R}^n$ $\quad \dim V = n$ because any basis of V have n elements

There exists infinite many basis of V, but all of them have the same dimension.

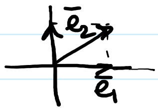
(See page 173 Lay / Theorem 4.5.10 page 244 Lay)

Def 5 $\quad \bar{u} \in V$, $\bar{u} = (x_1, x_2, \ldots x_n) = x_1\bar{e}_1 + \cdots + x_n\bar{e}_n$ $\quad B = \{\bar{e}_1, \ldots \bar{e}_n\}$

Def 5 $\quad \bar{u} \in V \quad , \quad \bar{u} = (x_1, x_2, \dots x_n) = x_1 \bar{e}_1 + \dots + x_n \bar{e}_n \quad B = \{\bar{e}_1, \dots \bar{e}_n\}$ base of V

The coordinates of a vector $\bar{u} \in V$ are the constants $x_1, x_2, \dots x_n$ of the linear combination $\bar{u} = x_1 \bar{e}_1 + \dots + x_n \bar{e}_n$ with respect to a chosen base B of V.

So:

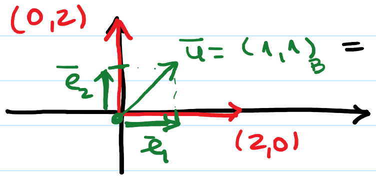


$V = R^2 \quad B = \{\bar{e}_1, \bar{e}_2\}$ standard base

$\bar{u} = (1,1) = 1\bar{e}_1 + 1\bar{e}_2$

Now if $C = \{\bar{w}_1, \bar{w}_2\}$ is another Base of $V = \mathbb{R}^2$

$\bar{w}_1 = (2,0) \quad , \quad \bar{w}_2 = (0,2) \quad ,$ for example , then



Def 6 : given $T: V \to W$

$\bar{u} \to T(\bar{u}) = A\bar{u}$

$$\text{Rank}(A) = \dim(\text{Col}(A))$$

The dimension of the Column space of A is called Rank of A

Rank Theorem _ §2.9 Theorem 14

For any matrix $A_{m \times n}$

$$\text{rank}(A) + \dim(\text{Null}(A)) = n$$

$$\left[\dim(\text{Col}(A)) + \dim(\text{Null}(A)) = n\right]$$