Föreläsning_5.3_alice

den 4 december 2020 11:47

Mål - • $\left\{\begin{array}{l} \mathrm{Ker}(T) \leftrightarrow \mathrm{Nul}\ A \\ \mathrm{Range}(T) \leftrightarrow \mathrm{Col}\ (A) \end{array}\right.$ $A =$ matrix of $T: V \rightarrow W$

page 221 Lay - chapter 4

- determinant
- Change of Basis Matrix & Chang of coordinates
- Eigenvalues & Eigenvectors

Still about F. 5.2

$T: V \longrightarrow W$ $V = \mathbb{R}^n$ $W = \mathbb{R}^m$ T linear transform

$\vec{u} \longrightarrow T(\vec{u}) = A_{m\times n}\,[u]_{n\times 1}$

$\mathrm{Ker}(T) = \{\bar{u} \in V \,/\, T(\bar{u}) = \bar{0}\}$ $\mathrm{Range}(T) = \{\bar{w} \in W \,/\, \bar{w} = T(\bar{u}) \text{ for some } \bar{u} \in V\}$

$\mathrm{Nul}(A) = \{x \in V \,/\, Ax = 0\}$ $\mathrm{Col}(A) = \mathrm{Span}\{[a_1], \ldots [a_n]\}$

$[a_i] \equiv$ columns of $A_{m\times n}$

$\mathrm{Nul}(A) \subseteq V$ subspace of V

$\mathrm{Nul}(A) = \{\bar{0}\} \Leftrightarrow T\ 1:1$

$\mathrm{Nul}(A) = \{\bar{0}\} \Leftrightarrow Ax = 0$ has ONLY the trivial solution

$\mathrm{Nul}(A_{n\times n}) = \{\bar{0}\} \Leftrightarrow \exists\ A^{-1}_{n\times n}$

$\mathrm{Nul}(A_{n\times n}) = \{\bar{0}\} \Leftrightarrow \det A_{n\times n} \neq 0$

Obs: $A.A^{-1} = I$

$\det(A.A^{-1}) = \det A . \det A^{-1} = \det I = 1$

$\Rightarrow \det A^{-1} = \dfrac{1}{\det A}$

$\mathrm{Col}(A) \subseteq W$ subspace of W

If $n \geq m \Rightarrow \mathrm{Col}(A) \supseteq W$ & T is ONTO

$\left[\begin{array}{l} \text{If } n = m, \ \& \ \{[a_1], \ldots [a_n]\} \text{ L I} \\ \text{Then } B = \{[a_1], [a_2] \ldots [a_n]\} \text{ is} \end{array}\right.$

a base of $\mathrm{Col}(A)$ & a base of W.

If $n < m$, then $\exists\ \vec{w} \in W, \vec{w} \notin \mathrm{Col}(A)$

so this means $Ax = w$ has No solution $\therefore$ T is NOT ONTO

$v \in W \cap \mathrm{Col}(A) \Rightarrow Ax = v$ is consistent.

Theorem

$$\left[\begin{array}{l} A_{m\times n} \\ \underbrace{\dim(\mathrm{Col}(A))}_{\overset{\|}{\mathrm{Rank}(A)}} + \dim(\mathrm{Nul}(A)) = n \end{array}\right.$$

Examples:

1) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$ $\quad \underbrace{[a_1]}\ [a_2]$

$Col(A) = Span\{[a_1], [a_2]\}$

$\{[a_1], [a_2]\}$ LI

$\Rightarrow B = \{[a_1][a_2]\}$ is a Base for $Col(A)$

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$(x,y) \to \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x+2y \\ 3y \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

<u>Obs</u> B is a base for $Col(A) \neq \mathbb{R}^3$

$\dim Col(A) = 2$

2) $A_1 = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

$Col(A_1) = Span\{[a_1], [a_2], [a_3]\}$

& $D = \{[a_1], [a_2], [a_3]\}$ LI

so D is a base for $Col(A_1)$

& since $\dim(Col(A)) = 3 = \dim(W = \mathbb{R}^3)$

D is also a base for $W = \mathbb{R}^3$.

3) $A_3 = \begin{array}{cccc} [a_1] & \dots & & [a_4] \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 2 \\ 3 \\ 0 \end{matrix} & \begin{matrix} 4 \\ 5 \\ 6 \end{matrix} & \left.\begin{matrix} 7 \\ 8 \\ 9 \end{matrix}\right] \end{array}$

$T: \mathbb{R}^4 \to \mathbb{R}^3$

$(x,y,z,t) \to \begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$

$Col(A) = Span\{[a_1], [a_2], [a_3], [a_4]\}$

$\mathcal{E} = \{\underbrace{[a_1], [a_2]\ [a_3]}_{D}, [a_4]\}$ is LD but $\overbrace{D \subseteq \mathcal{E}}$ is LI

So is is possible to extract a Base for $Col(A)$ from its set of generators $\mathcal{E}$.

4) $V = Span\{\bar{v}_1, \bar{v}_2\}$ $\quad \bar{v}_1 = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$ $\quad \bar{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

•) Obtain Base for V & dim V

•) $\bar{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \in V$?

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Solution : • To obtain a base for V, we have to verify if the set of its generators $\{\bar{v}_1, \bar{v}_2\}$ is L I.

- To ask if $\bar{w} \in V$, is to verify if $\exists\, \alpha, \beta$ such that $\bar{w} = \alpha \bar{v}_1 + \beta \bar{v}_2$ ($\bar{w}$ is linear combination of $\bar{v}_1$ & $\bar{v}_2$)

So we can solve these two questions at the same time computing the Echelon form of $[\bar{v}_1 \bar{v}_2 | w]$

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -1 & 3 \\ 0 & 2 & 6 \\ 2-\frac{2}{3}.3 & 1-\frac{2}{3}(-1) & 7-\frac{2}{3}(3) \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -1 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \to$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} 1\alpha + 0\beta = 2 \\ 0\alpha + 1\beta = 3 \\ 0 \quad\quad = 0 \end{cases} \Rightarrow \begin{matrix} \alpha = 2 \\ \beta = 3 \end{matrix} \quad \text{unique solution.}$$

So $\bar{w} \in V$ $\quad \bar{w} = 2\bar{v}_1 + 3\bar{v}_2$ $\quad [\bar{w}]_B = (2,3)_B$

where $B = \{\bar{v}_1, \bar{v}_2\}$ base of $V = \text{Span}\{\bar{v}_1, \bar{v}_2\}$.

$[\bar{w}]_B$ are called the Coordinates of $\bar{w}$ with respect to the Base B.

Obs. • when the Base is no longer the Canonical ≡ Standard

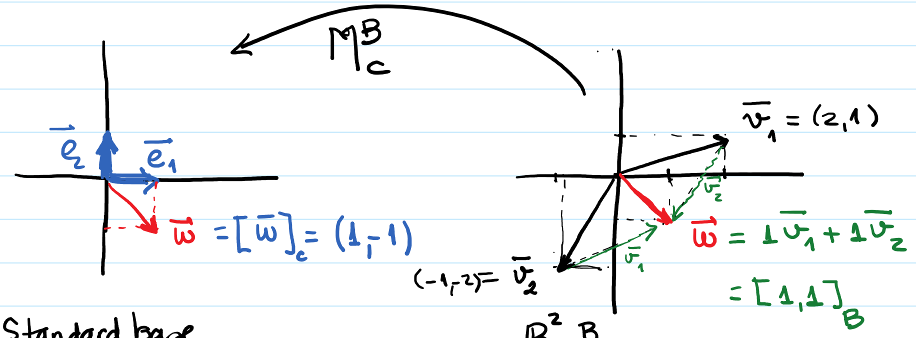
• we will denote $[\bar{w}]_B \leftarrow$ the index B to clarify the reference system (new Base)

Change of Base - Matrix

Change of Coordinates Matrix Lay page 237.

We will start with an example.

$\mathbb{R}^2, C$



$C = \{\bar e_1, \bar e_2\}$ Standard base
(or Canonical Base)
$\bar e_1 = (1,0)$ $\bar e_2 = (0,1)$

1) $\vec w$ is the same vector on both reference systems C & B

$$[1,-1]_C = \vec w = [1,1]_B$$

$\mathbb{R}^2, B$

$B = \{\bar v_1, \bar v_2\}$

Obs $\bar v_1 = 1\vec v_1 + 0\vec v_2 = (1,0)_B$
$\vec v_2 = 0\vec v_1 + 1\vec v_2 = (0,1)_B$

$B = \{\vec v_1, \vec v_2\}$ the vectors $\bar v_1$ & $\bar v_2$ are the reference inside B

So, $\bar w$ has different coordinates, depending on the base.

Question. How to go from one set of coordinates to the other?

Answer. Using the matrix of change of Basis

So Since $B = \{\bar v_1, \bar v_2\}$ is a base of $V = \mathbb{R}^2$

$$(\alpha,\beta)_B = \bar w = \alpha \bar v_1 + \beta \bar v_2 = \alpha(2\bar e_1 + 1\bar e_2) + \beta(-1\vec e_1 - 2\vec e_2) =$$

$$= (2\alpha-\beta)\vec e_1 + (1\alpha - 2\beta)\vec e_2$$

$$\|$$

$$(x,y)_C = \bar w = x\,\bar e_1 + y\,\vec e_2$$

Now Comparing both formulations:

$$(2\alpha-\beta)\vec e_1 + (1\alpha+2\beta)\vec e_2 = w = x\bar e_1 + y\bar e_2$$

$$\begin{cases} 2\alpha - \beta = x \\ 1\alpha - 2\beta = y \end{cases} \Leftrightarrow \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Obs:

$$\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Obs:

$$\begin{array}{cc} 2 & -1 \\ 1 & -2 \\ \uparrow & \uparrow \\ [\bar{v}_1]_C & [\bar{v}_2]_C \end{array} \quad \underbrace{\begin{bmatrix}\alpha\\ \beta\end{bmatrix}}_{\updownarrow \atop [\bar{w}]_B} = \underbrace{\begin{bmatrix}x\\ y\end{bmatrix}}_{\updownarrow \atop [\bar{w}]_C}$$

So $\left[[\vec{v}_1]_c \ [\vec{v}_2]_c \right] = M_c^B \equiv$ matrix of the change of Basis from base B to base C (Here C = canonical/Standard Base)

So $\begin{bmatrix}2 & -1\\ 1 & -2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \underbrace{\begin{bmatrix}1\\ -1\end{bmatrix}}$

$$M_C^{B} \, [\bar{w}]_{B} = [w]_C$$

Computing M^{-1} we have $M^{-1} = M_B^C$

$$\underbrace{M^{-1}\cdot M}_{I} \, [w]_B = M^{-1}[w]_c$$

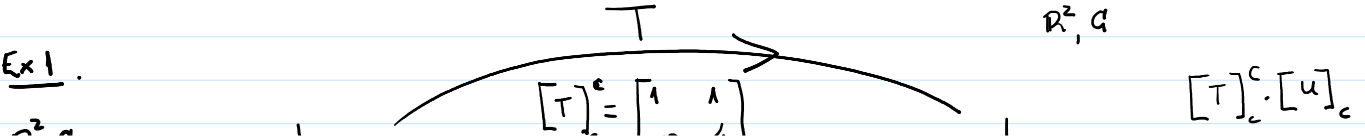
$$\boxed{[w]_B = M^{-1}[w]_c}$$

$$M^{-1} = \frac{1}{\det M}\begin{pmatrix}-2 & 1\\ -1 & 2\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}-2 & 1\\ -1 & 2\end{pmatrix} = \begin{pmatrix}2/3 & -1/3\\ 1/3 & -2/3\end{pmatrix}$$

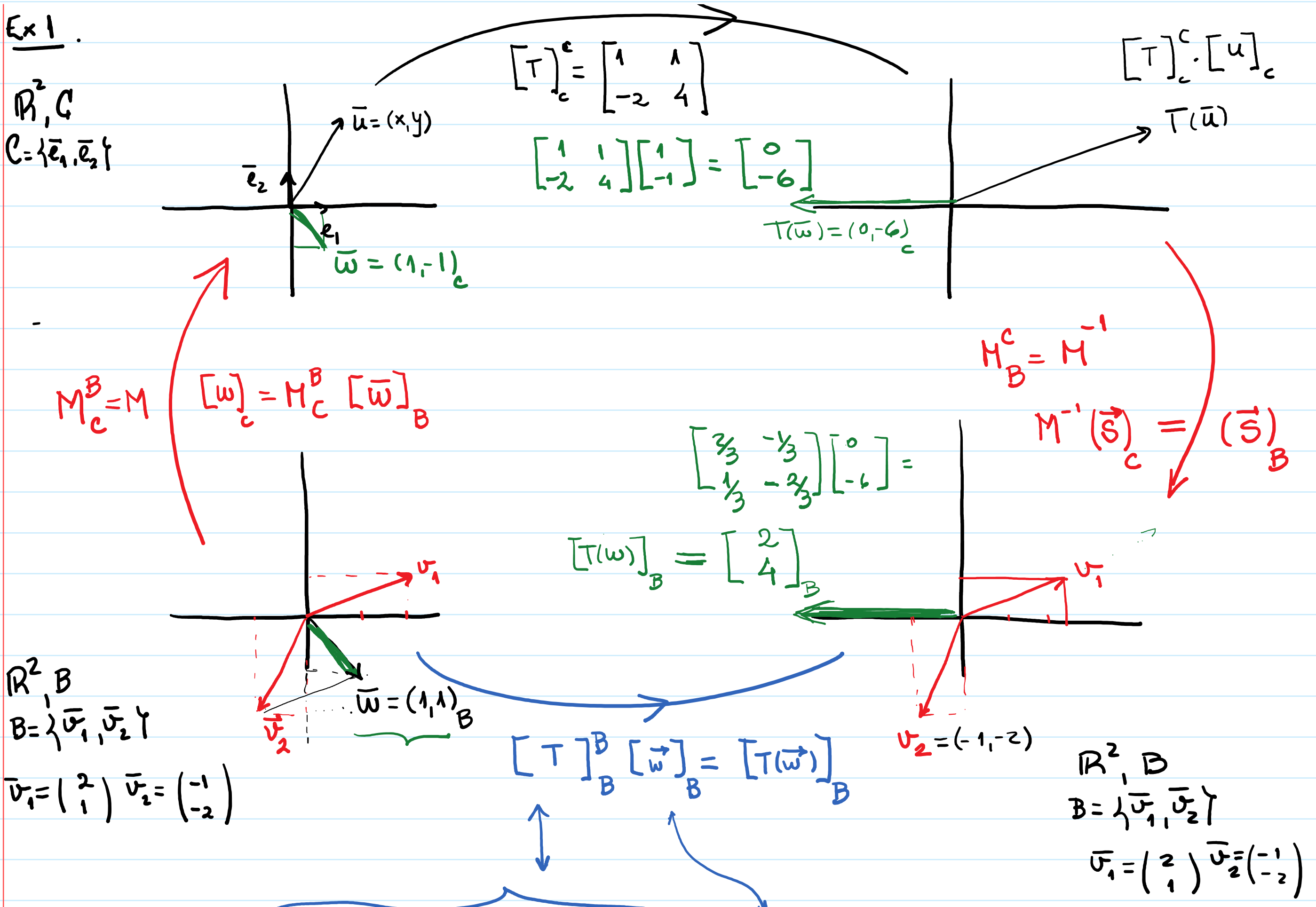
$$M^{-1} = M_B^C \quad \therefore \quad \begin{pmatrix}2/3 & -1/3\\ 1/3 & -2/3\end{pmatrix}\underbrace{\begin{pmatrix}1\\ -1\end{pmatrix}}_{[\bar{w}]_c} = \begin{pmatrix}2/3+1/3\\ 1/3+2/3\end{pmatrix} = \underbrace{\begin{pmatrix}1\\ 1\end{pmatrix}}_{[\bar{w}]_B}$$

☺k! Now we are going to see how different basis from the spaces V & W affect the formulation of the matrix associated to the linear Transform $T: V \to W$.

Ex 1.



Ex 1.



$$[T]^B_B\,[\vec w]_B=[T(\vec w)]_B$$

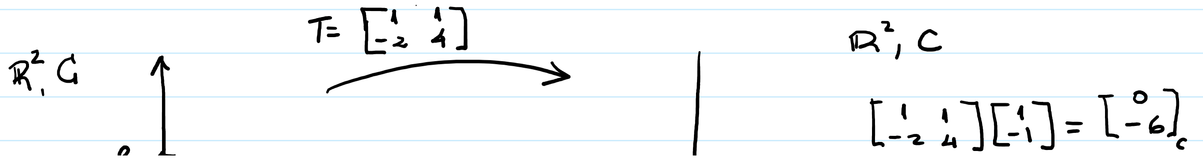
$$\updownarrow$$

$$\underbrace{-\frac{1}{3}\begin{bmatrix}-2&1\\-1&2\end{bmatrix}\underbrace{\begin{bmatrix}1&1\\-2&4\end{bmatrix}\underbrace{\begin{bmatrix}2&-1\\1&-2\end{bmatrix}[\bar w]_B}_{[w]_C}}_{[T(\bar w)]_C}}_{[T(w)]_B}=[T(\bar w)]_B$$

So
$$[T]^B_B=\begin{bmatrix}2&0\\1&3\end{bmatrix}$$

This is the matrix of T with respect to the Base B, assuming it as a reference in V and in W.

Ex2 Now I will repeat the same scheme, for the same $T:\mathbb{R}^2\to\mathbb{R}^2$ But assuming another Base $K=\{\bar u_1,\bar u_2\}$ K is going to be a very special Base.

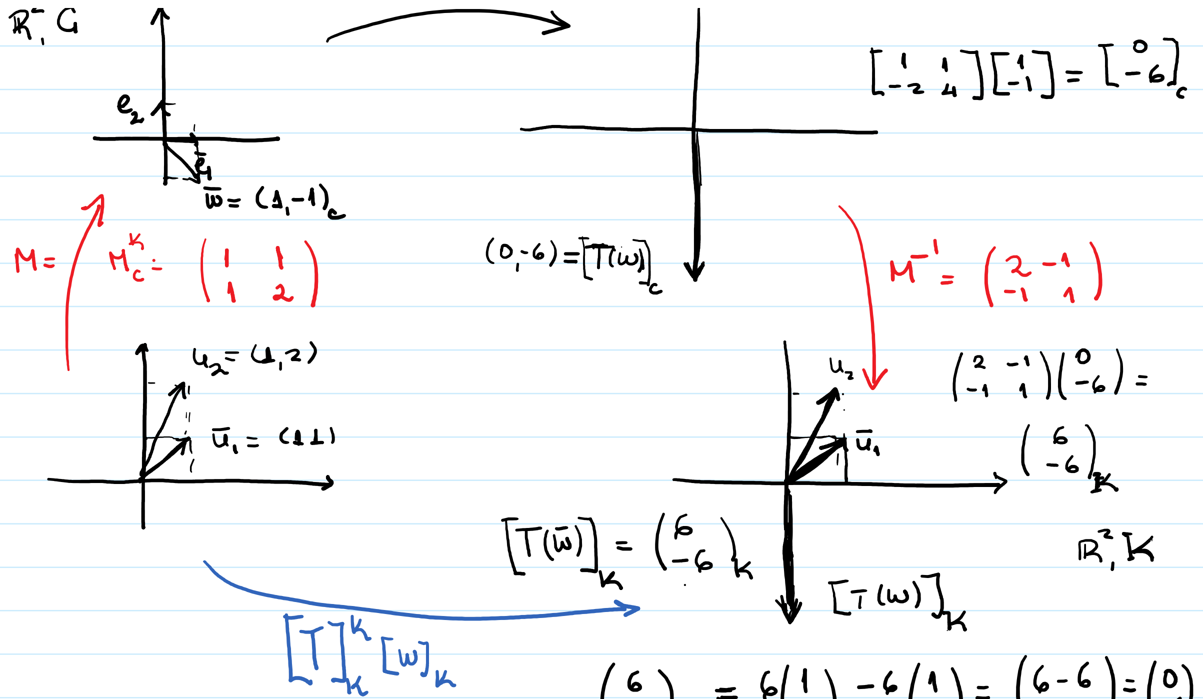


$\mathbb{R}^2, \underline{K}$

$\underline{K} = \{\bar{u}_1, \bar{u}_2\}$

$\bar{u}_1 = \begin{pmatrix}1\\1\end{pmatrix}$

$\bar{u}_2 = \begin{pmatrix}1\\2\end{pmatrix}$



$M = M^K_c = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$

$\begin{bmatrix}1 & 1\\ -2 & 4\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} = \begin{bmatrix}0\\ -6\end{bmatrix}_c$

$M^{-1} = \begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$

$\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0\\ -6\end{pmatrix} = \begin{pmatrix}6\\ -6\end{pmatrix}_K$

$[T(\bar{w})]_K = \begin{pmatrix}6\\ -6\end{pmatrix}_K$

$$\begin{pmatrix}6\\ -6\end{pmatrix}_K = 6\underbrace{\begin{pmatrix}1\\ 1\end{pmatrix}}_{\bar{u}_1} - 6\underbrace{\begin{pmatrix}1\\ 2\end{pmatrix}}_{u_2} = \begin{pmatrix}6-6\\ 6-12\end{pmatrix} = \begin{pmatrix}0\\ -6\end{pmatrix}_c$$

$$[T]^K_K = M^{-1} [T]^c_c [M]^K_C$$

$$\underbrace{\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\underbrace{\begin{bmatrix}1 & 1\\ -2 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}}}$$

$$\underbrace{\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\ 2 & 6\end{bmatrix}}$$

$$\begin{bmatrix}\textcircled{2} & 0\\ 0 & \textcircled{3}\end{bmatrix} \longleftarrow \text{diagonal Matrix!}$$

<u>How Nice!</u>

Why does it happen?

Is it allways possible to find a base K such that

$$[T]^K_K = \begin{bmatrix}\alpha_1 & & 0\\ & \ddots & \\ 0 & & \alpha_n\end{bmatrix} \text{ diagonal matrix?}$$

Take a look in one more <u>detail</u>:

$$T(1,1) = \begin{bmatrix}1 & 1\\ -2 & 4\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}1+1\\ -2+4\end{bmatrix} = \begin{bmatrix}2\\ 2\end{bmatrix} = \textcircled{2}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$T(5,5) = \begin{bmatrix}1 & 1\\ -2 & 4\end{bmatrix}\begin{bmatrix}5\\ 5\end{bmatrix} = \begin{bmatrix}5+5\\ -10+20\end{bmatrix} = \begin{bmatrix}10\\ 10\end{bmatrix} = \textcircled{2}\begin{bmatrix}5\\ 5\end{bmatrix}$$

$$T(1,2) = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\end{bmatrix} = \begin{bmatrix}1+2\end{bmatrix} = \begin{bmatrix}3\end{bmatrix} = \textcircled{3}\begin{bmatrix}1\end{bmatrix}$$

$$T(1,2) = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -2+8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \textcircled{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(10,20) = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 10+20 \\ -20+80 \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \end{bmatrix} = \textcircled{3} \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

So $\boxed{T(\bar{w}) = \lambda w}$ $\left\{ \begin{array}{l} \lambda_1 = 2 \ , \ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_2 = 3 \ , \ v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{array} \right.$

$\uparrow$

When this is satisfied, λ is called eigenvalue of T and the corresponding v is called eigenvector (associated to λ)

⊛ When we can compute a base formed by eigenvectors of a Matrix, This matrix has a diagonal form when written in terms of this base.

⊛ Not allways it is possible, But if T is symmetric, this is allways True.

⊛ For the applications involving systems of ODE, finding Base of eigenvectors and Transforming the matrix of the system in a diagonal matrix is a crucial part of the solution method.

So, Now the question is: How to compute

eigenvalues & eigenvectors !.

Please Check now the file for Föreläsning_5.3.pdf from the previous teacher. or Lay Book, Chapter 5.

Of course on monday I will talk about that.

I hope this motivation will help to understand the diagonalization method presented on monday for solving systems of ODE.

Happy weekend ! ☺