Föreläsning_6.1_alice

den 7 december 2020 07:58

Goal:
- Eigenvalues & Eigenvectors
- Determinant
- Diagonalisation $A = M^{-1} D M \quad D = \begin{pmatrix} \lambda_1 & \\ & \ddots \\ & & \lambda_n \end{pmatrix}$
- solution of systems of ODE

I) $T: V \longrightarrow V$ $\quad V = \mathbb{R}^n$, domain = Codomain

$u \longrightarrow T(u)$ $\quad T$ linear Transf.

$T = [T] = A_{n\times n}$ (coeff given according to the standard Basis $\{\vec{e}_1, \dots \vec{e}_n\}$, $\vec{e}_1 = (1, 0, \dots 0)$)

Def: If $\exists$ (there exists) $\lambda \neq 0$, $\vec{v} \in V$ suchthat ($\vec{v} \neq \vec{0}$)

$$\boxed{T(\vec{v}) = \lambda \vec{v}}$$

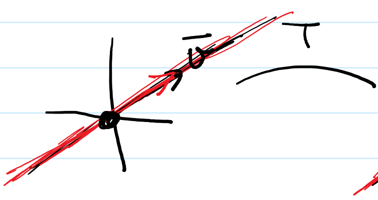
$T(\vec{0}) = \vec{0}$

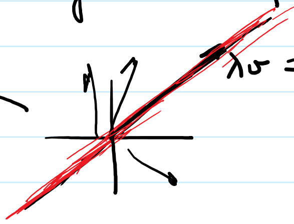
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Then λ is called Eigenvalue of T
& v is called Eigevector of T (associated to λ)

$I_{n\times n} X_{n\times 1} = X$

$I_{n\times n} \cdot A = A$





$\lambda v = T(\vec{v})$

Obs: if $\vec{v}$ is an eigenvector of T associated to λ, all the vector on the $\text{Span}\{\vec{v}\} = \{\alpha\vec{v}, \alpha \in \mathbb{R}\}$ are also eigenvectors from T associated to λ.

Proof: $T(\alpha\vec{v}) = \alpha T(\vec{v}) = \alpha(\lambda\vec{v}) = \lambda(\alpha\vec{v})$

II) How to compute $\lambda, \vec{v}$ $\quad \boxed{T(\vec{v}) = \lambda\vec{v}}$ $\quad T: V_c \to V_c$ $\quad A_{n\times n}$ because $T: V \to V$

$T(\vec{v}) = [T]_c^c [\vec{v}]_c = [T][\vec{v}]$ (coordinates given according to the standard Base)

$[T] = A_{n\times n}$ square matrix

$[v] = X$

(Obs $I_{n\times n} X_{n\times 1} = X_{n\times 1}$

$[v] = x$

$\boxed{T(v) = \lambda v} \Leftrightarrow \boxed{Ax = \lambda x} \Leftrightarrow \begin{cases} \text{Obs } I_{n\times n} x_{n\times 1} = x_{n\times 1} \\ Ax = \lambda I x \end{cases} \Leftrightarrow$

$\Leftrightarrow \quad Ax - \lambda I x = \vec{0} \quad \Leftrightarrow \quad \boxed{(A - \lambda I) x = \vec{0}}$

We want to find Non trivial solutions for the new system:

$(A - \lambda I) x = \vec{0}$

This means: $\det(A - \lambda I) = 0$ and λ are the values that are going to satisfy this condition

$T: V \to V$

$v \to T(v) = \overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}}^{A} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

Ex: $\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{T(u)} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 1y \\ 1x + 2y \end{bmatrix}$

looking at λ, v (eigenvalue / eigenvectors) $\boxed{T(\bar{v}) = \lambda \bar{v}}$

$\boxed{(A - \lambda I)\bar{x} = \bar{0}}$

$$\Bigg(\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{(A - \lambda I)}\Bigg) \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\vec{0}}$$

So, we are looking for Non Trivial solutions

We have to impose that $\boxed{\det(A - \lambda I) = 0}$

$$\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$\boxed{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc}$

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = 0$$

$p(\lambda) = \det(A - \lambda I)$ characteristic polynomial

$$p(\lambda) = (2-\lambda)(2-\lambda) - 1 \cdot 1 = 0$$

λ are going to be the roots of $p(\lambda) = 0$

$$p(\lambda) = 4 - 2\lambda - 2\lambda + \lambda^2 - 1$$

$$p(\lambda) = \lambda^2 - 4\lambda + 3$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 3}}{2}$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{4}}{2} \quad \frac{4+2}{2} = \frac{6}{2} = 3 \qquad \frac{4-2}{2} = \frac{2}{2} = 1$$

$$\lambda_1 = 3 \qquad \lambda_2 = 1$$

Now we are going to compute the eigenvectors associated to $\lambda_1 = 3$ / $\lambda_2 = 1$

$\boxed{\lambda_1 = 3} \Rightarrow (A - 3I)X = \vec{0}$ our eigenvectors are the solution of this system.

$$\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-3 & 1-0 \\ 1-0 & 2-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) L_2 \leftarrow L_2 + L_1$$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow -x + 1y = 0 \Rightarrow \boxed{x = y}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \alpha \in \mathbb{R}$$

$\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector associated to $\lambda_1 = 3$

$$\det(A-\lambda I)=0 \qquad p(\lambda)$$

$$(A-\lambda I)X=0$$

$$\boxed{\lambda=1} \Rightarrow \begin{pmatrix}2-1 & 1\\ 1 & 2-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

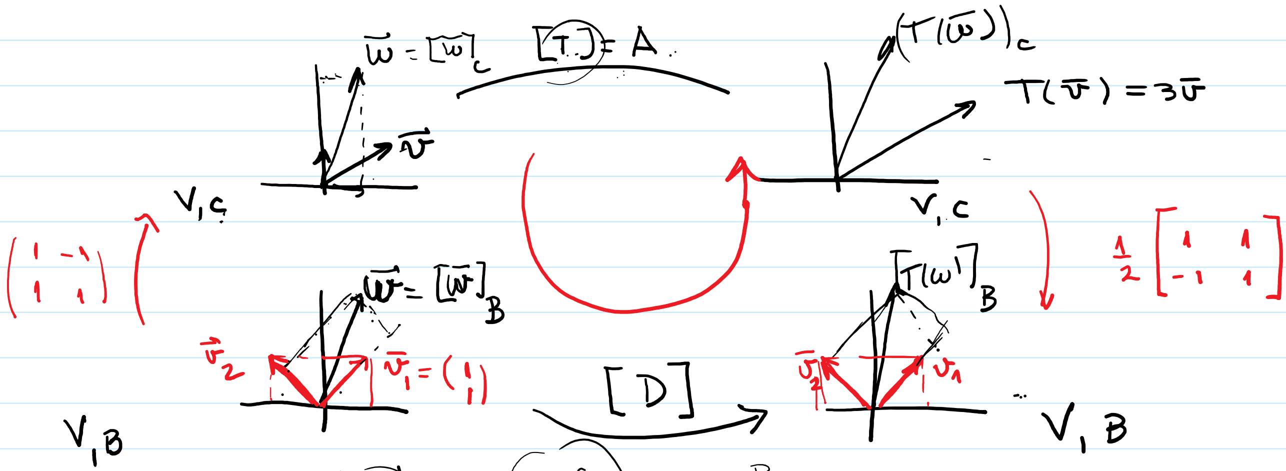
$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \left(\begin{array}{cc|c}1 & 1 & 0\\ 1 & 1 & 0\end{array}\right) \Rightarrow \left(\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right) \Rightarrow$$

$$\boxed{x+y=0 \Rightarrow \boxed{x=-y}} \Rightarrow \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y\\y\end{pmatrix}=y\begin{pmatrix}-1\\1\end{pmatrix}=\beta\begin{pmatrix}-1\\1\end{pmatrix};\ \beta\in\mathbb{R}$$

$$\lambda_2=1 \quad \vec{v}_2=\begin{pmatrix}-1\\1\end{pmatrix}$$

So $A=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \longrightarrow M^B_C=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$ $\quad V=\mathbb{R}^2$

$$M^C_B=\frac{1}{2}\begin{pmatrix}1 & +1\\ -1 & 1\end{pmatrix}$$



$$\underset{M^{-1}}{\frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}}\ \underset{A^C_C}{\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}}\ \underset{M^B_C}{\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}}$$

$M=M^B_C$

$M^{-1}=M^C_B$

$$\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}3 & -1\\ 3 & 1\end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix}6 & 0\\ 0 & 2\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}}_{D}$$

$D=[T]^B_B$

$M=M^B_C$

$$\underbrace{\qquad}_{D} \qquad\qquad _B$$

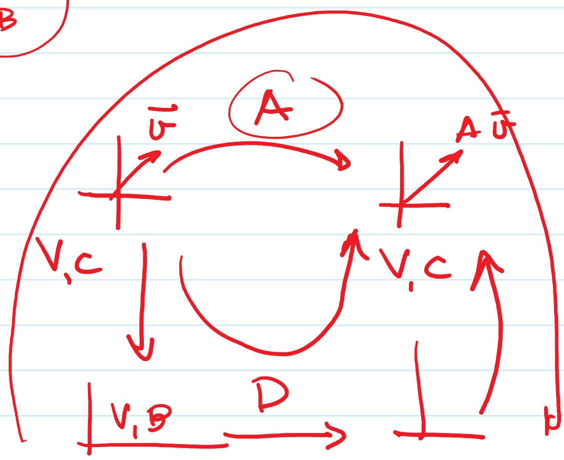
$M = M_C^B$

$M^{-1} = M_B^C$

$$\boxed{M\,[D]_B^B = \underbrace{M\,M^{-1}}_{I}\,[A]_C^C\,[M]_C^B}$$

$$M_C^B\,[D]_B^B\,M_B^C = [A]_C^C\,\underbrace{M_C^B \cdot M_B^C}_{I}$$

$$M_C^B\,[D]\,M_B^C = [A]_C^C$$



One Application : Solving systems of ODE

$$x^{(4)} - 7x^{(3)} + 4x'' + 5x' - 2x = 0 \qquad \text{4th order ODE}$$

Goal : Transf this eq into a system. $\boxed{X' = AX}$

$$\begin{cases} \boxed{x_1 = x} \\ x_2 = x' = x_1' \\ x_3 = x'' = x_1'' = x_2' \\ x_4 = x''' = x_1''' = x_2'' = x_3' \\ \boxed{x_4' = (x''')'} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \to \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix}$$

$\boxed{x_1' = a\,x_1}$

$\boxed{x_2' = b\,x_2}$

$\boxed{x_4' = c\,x_4}$

$$(x''')' = x^{(4)} = +2x_1 - 5x_2 - 4x_3 + 7x_4$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -5 & -4 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\to \begin{pmatrix} \vdots \\ x_4' \end{pmatrix} \quad \begin{pmatrix} 2 & -5 & -4 & 7 \end{pmatrix} \begin{pmatrix} \\ x_4 \end{pmatrix}$$

$$\boxed{x' = A \cdot x} \qquad \boxed{y' = a y}$$

ODE

$\exp(A)$

$$\boxed{y(t) = c_1 e^{at}}$$

system solution : $\boxed{X = C \exp(tA)}$???

Obs. $A \to M^{-1} D M$

$$A^2 = \underbrace{(M^{-1} D M)(M^{-1} D M)}_{I} = M^{-1}(D^2)M$$

$$A^3 = (M^{-1} D^2 M)(M^{-1} D M) = M^{-1} D^3 M$$

$$\boxed{A^n = M^{-1} D^n M}$$

$$\exp(t) \cong \sum_{k=0}^{n} \frac{t^k}{k!} \quad \text{Taylor polinomial}$$

$$\exp(At) \cong \sum_{k=0}^{n} \frac{(tA)^k}{k!} = \sum \frac{t^k A^k}{k!} \quad \circledast$$

$$= M^{-1} \left(\sum_{k=0}^{n} \frac{t^k D^k}{k!} \right) M$$

Ex2 IVP
$$\begin{cases} x'' - 3x' + 2x = \cos 3t \\ x(0) = 2 \\ x'(0) = -3 \end{cases}$$

$$\to x'' = -2x + 3x' + \cos 3t \qquad x_1 = x$$

$$x'' = -2x + 3x' + \underbrace{\cos 3t}$$

$x_1 = x$

$x_2 = x' = x_1'$

$x_2' = x''$

$x_1' = 0x_1 + 1x_2$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos 3t \end{pmatrix}$$

font

$$\begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longleftrightarrow \text{ if } \begin{array}{l} x_1' = a_1 x_1 \Rightarrow \\ x_2' = a_2 x_2 \Rightarrow \end{array}$$

$\updownarrow$ D diagonalizing A, to obtain $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

Formulation presented (F 6.1) previous teacher.

$$\boxed{X'(t) = A_c\, X(t)}$$

$A_{n \times n}$

$n = 2$

$(v_1)\ (v_2)$

$n = n \ldots$

$(v_1) \ldots (v_n)$

$$M_B^c X'(t) = M_B^c A I X(t)$$

$$\underbrace{M^{-1} X'(t)}_{(X')_B} = \underbrace{M^{-1} A M}_{D} \underbrace{M^{-1} X(t)}_{(X)_B}$$

$(X')_B = Y'$ $\qquad$ $(X)_B = Y$

$\leftarrow$ Refence system is the eigenvector Base.

$$y' = D y$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}' = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\left\{ y_1' = \lambda_1 y_1 \right. \rightarrow y_1 = c_1 e^{\lambda_1 t} \qquad \begin{pmatrix} y_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \end{pmatrix}$$

$$\begin{cases} y_1' = \lambda_1 y_1 \\ \vdots \\ y_n' = \lambda_n y_n \end{cases} \quad \begin{array}{l} \to y_1 = c_1 e^{\lambda_1 t} \\ \vdots \\ \to y_n = c_n e^{\lambda_n t} \end{array}$$

$$\underbrace{\begin{pmatrix} y_1 \\ | \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}}_{\text{in the Base } B}$$

$$X = M_c^B Y = \underbrace{\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix}}_{\text{matrix with the eigenvectors}} \quad n \in \mathbb{N}.$$

$$\boxed{X = c_1 (v_1) e^{\lambda_1 t} + c_2 (v_2) e^{\lambda_2 t} + \dots + c_n (v_n) e^{\lambda_2 t}}$$

$n = 2$ $\quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; v_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$X'(t) = A\,X(t) \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

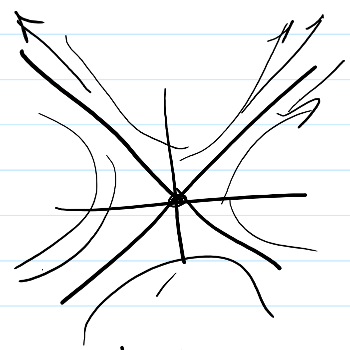
$$X' = AX$$

$$\underbrace{M^{-1} X'}_{} = \underbrace{M^{-1} A M}_{} (M^{-1} X)$$

$$y' = Dy \Longrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{1t} \end{pmatrix}$$

$$\boxed{X = M_c^B (y)} \qquad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{1t} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{1t}}$$



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{1t} \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} - c_2 e^{1t} \\ c_1 e^{3t} + c_2 e^{1t} \end{pmatrix} =$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{1t}$$