Föreläsning_6.2_alice

den 10 december 2020 07:23

Goal: $\begin{cases} \text{Eigenvalues / Eigenvectors / Complex eigenvalues} \leftarrow \\ \text{Diagonalisation} \end{cases}$

Applications: $\begin{cases} \text{Solution of } X' = AX \\ A^n \end{cases}$

Theo 5 page 300 §5 - Lay.

$$\left[A_{n\times n} \text{ is diagonalizable} \right] \Longleftrightarrow \left[A_{n\times n} \text{ has } n \text{ LI eigenvectors} \right]$$

Obs. $A_{n\times n}$: $T: V \to V$

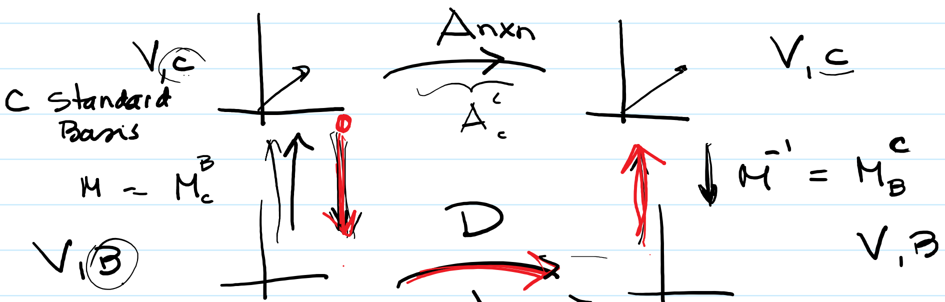
$u \to T(u) = A_{n\times n} [u]_{n\times 1}$

$\dim V = n$

Any set of LI vectors with n elements forms a basis for the space V.

$$\left[A \text{ diagonalisable} \right] \Longleftrightarrow \left[\begin{array}{l} B = \{\bar{v}_1, \dots, \bar{v}_n\} \quad Av_i = \lambda_i v_i \\ B \text{ a basis for } V. \end{array} \right]$$

Recal.



B = basis of eigenvectors

$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$Av_i = \lambda_i v_i$

$$D = M^{-1} A_{n\times n} M_c^B \leftarrow$$

$$D = M^{-1} A_{n\times n} M_C^B$$

$$A = \underbrace{M_C^B}\, \underbrace{D \quad M^{-1}}$$

Application : $\boxed{X' = A X}$ solution of ODE-systems.

Ex :

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -2 & 4 \\ 3 & -4 & 4 \\ 3 & -2 & 2 \end{pmatrix}}_{A_{3\times3}} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \qquad \boxed{X' = A X}$$

Solution using Eigenvectors :

Steps : 1) Find eigenvalues of $A_{3\times3}$

$$\det(A - \lambda I) = 0 \qquad (p(\lambda) = 0)$$

2) Find eigenvalues $\boxed{(A - \lambda I) v = 0}$

$$M = M_C^B = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \qquad D = \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}}$$

$$M^{-1} = M_B^C = (M_C^B)^{-1}$$

3) $X' = A X$ (C = canonical standard basis)

$$\underbrace{M^{-1}}_{M_B^C} X'_c = M^{-1} A X.$$

$$= \left(\underbrace{M^{-1} A M_C^B}\right) \underbrace{\left(M^{-1} X\right)}_{Y}$$

I

$$\boxed{Y' = D Y}$$

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 \; y_1(t) \\ \lambda_2 \; y_2(t) \\ \lambda_3 \; y_3(t) \end{pmatrix}$$

$$y_1(t) = c_1 e^{\lambda_1 t} \qquad (c_1 e^{\lambda_1 t}$$

$$y_1(t) = c_1 e^{\lambda_1 t}$$
$$y_2(t) = c_2 e^{\lambda_2 t} \qquad y = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_3 t} \end{pmatrix}$$
$$y_3(t) = c_3 e^{\lambda_3 t}$$

4) $M^{-1} X = Y \iff X = M_C^B Y$

$$\boxed{X = c_1 (v_1) e^{\lambda_1 t} + c_2 (v_2) e^{\lambda_2 t} + c_3 (v_3) e^{\lambda_3 t}}$$

5) if IVP $\begin{cases} \text{system:} \\ X(0) = X_0 \end{cases}$ $\Rightarrow$ we can compute the constants c_1, c_2, c_3.

Step 1: $\det(A - \lambda I) = 0$ $\qquad \boxed{Av = \lambda v}$ $\quad Av = \lambda I v$

$(A - \lambda I)v = 0$

$$\det \begin{pmatrix} 1-\lambda & -2 & 4 \\ 3 & -4-\lambda & 4 \\ 3 & -2 & 2-\lambda \end{pmatrix} = 0$$

1° way to compute:

$$\det(A - \lambda I) = (-1)^{1+1} (1-\lambda) \begin{vmatrix} -4-\lambda & 4 \\ -2 & 2-\lambda \end{vmatrix} + (-1)^{1+2} (-2) \begin{vmatrix} 3 & 4 \\ 3 & 2-\lambda \end{vmatrix}$$
$$+ (-1)^{1+3} \cdot 4 \begin{vmatrix} 3 & -4-\lambda \\ 3 & -2 \end{vmatrix}$$

OBS: this is full development ... trycky to solve.

2°. way to solve is using properties of $\det A = |A|$

1) $\det(A \cdot B) = \det A \cdot \det B$ $\qquad$ 3) $\det A = \det A^T$

2) $\det(A \cdot A^{-1}) = \det(I)$
$\det(A) \cdot \det A^{-1} = 1 \Rightarrow \det A^{-1} = \dfrac{1}{\det A}$

4) $A = \begin{bmatrix} - L_1 - \\ - L_2 - \end{bmatrix}$ $\qquad \tilde{A} = \begin{bmatrix} L_1 \\ \to cL_j \end{bmatrix}$ $\leftarrow$ just 1 row.

det A

4) $A = \begin{bmatrix} - L_1 - \\ - L_2 - \\ \vdots \\ - L_n - \end{bmatrix}$ $\tilde{A} = \begin{bmatrix} L_1 \\ \to c L_j \\ L_n \end{bmatrix}$ ← just 1 row.

$\det \tilde{A} = c \det A$ $\tilde{A} = cA = \begin{bmatrix} cL_1 \\ \vdots \\ cL_n \end{bmatrix} \Rightarrow \det \tilde{A} = c^n \det A$

5) $A = \begin{bmatrix} - L_1 \\ \vdots \\ - L_n \end{bmatrix}$ $\tilde{A} = \begin{bmatrix} L_1 \\ \vdots \\ L_i - \alpha L_j \\ \vdots \\ L_n \end{bmatrix}$ elementary operation $\cdot$(cte), +

$\det \tilde{A} = \det A$

6) $A = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$ $\tilde{A}$ = A with $L_i \leftrightarrow L_j$
exchange of 2 rows

Then $\det(\tilde{A}) = (-1) \det A$.

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ $(-1)\cdot$ $(-1)\cdot$ * Book.

Example:

$\begin{pmatrix} 1-\lambda & -2 & 4 \\ 3 & -4-\lambda & 4 \\ 3 & -2 & 2-\lambda \end{pmatrix}$ $L_3 \leftarrow L_3 - L_2$

$\begin{pmatrix} 1-\lambda & -2 & 4 \\ 3 & -4-\lambda & 4 \\ 0 & 2+\lambda & -2-\lambda \end{pmatrix}$ $C_2 \leftarrow C_2 + C_3$ $\left(\begin{matrix} \det A = \\ \det A^T \end{matrix} \right)$

$\begin{pmatrix} 1-\lambda & 2 & 4 \\ 3 & -\lambda & 4 \\ 0 & 0 & -2-\lambda \end{pmatrix}$

Now we can compute $\det(A - \lambda I) = 0$

considering the expansion with respect to Row 3.

$\det(A - \lambda I) = (-1)^{3+1} \cdot 0 \cdot |\quad| + (-1)^{3+2} \cdot 0 \cdot |\quad| + (-1)^{3+3} \cdot (-2-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{vmatrix}$

$$\det(A-\lambda I) = -(\lambda+2)\Big((-1-\lambda)(-\lambda) - 6\Big)$$

$$= -(\lambda+2)\left(\lambda^2-\lambda-6\right) \qquad \begin{cases} \lambda_1 = -2 \\ \lambda_2 = 3 \end{cases}$$

$$= -(\lambda+2)\left((\lambda+2)(\lambda-3)\right)$$

$$= -(\lambda+2)^2 \cdot (\lambda-3)$$

So $\underbrace{\lambda_1=-2, \quad \lambda_2=-2}$, $\lambda_3 = 3$

2) Eigenvectors:

$v_{\lambda=-2}$
$$\begin{pmatrix} 1-(-2) & -2 & 4 \\ 3 & -4-(-2) & 4 \\ 3 & -2 & 2-(-2) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} 0\ 0\ 0 \\ 0\ 0\ 0 \end{matrix} = \begin{matrix} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - L_1 \end{matrix} \begin{pmatrix} 3 & -2 & 4 \\ 3 & -2 & 4 \\ 3 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x - 2y + 4z = 0 \Rightarrow$$

$$3x = +2y - 4z \Rightarrow \boxed{x = \frac{2}{3}y - \frac{4}{3}z}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{3}y - \frac{4}{3}z \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{3}y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{4}{3}z \\ 0 \\ z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overset{\alpha}{\cancel{y}}\begin{pmatrix} 2/3 \\ 1 \\ 0 \end{pmatrix} + \overset{\beta}{\cancel{z}}\begin{pmatrix} -4/3 \\ 0 \\ 1 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}$$

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$$

$v_{\lambda=3}$
$$\begin{pmatrix} 1-3 & -2 & 4 \\ 3 & -4-3 & 4 \\ 3 & -2 & 2-3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 & 4 \\ 3 & -7 & 4 \\ 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

⋮ gaussian elimination

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x + 0 + z = 0 \quad \Rightarrow \quad x = z$$
$$/ \quad y - z = 0 \qquad\qquad y = z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = \overset{\alpha}{\cancel{z}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}$$

$$v_{\lambda=3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$M_c^B = \begin{pmatrix} 2 & -4 & 1 \\ 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} -2 & & \\ & -2 & \\ & & 3 \end{pmatrix}$$

or

$$M_c^B = \begin{pmatrix} 1 & 2 & -4 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & & \\ & -2 & \\ & & -2 \end{pmatrix}$$

<u>Application</u>: $A^8 = ?$ $\quad A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$

<u>Solution</u>: Diagonalise A and use the formulation involving its Diagonal form.

$$M = M_c^B \qquad M^{-1} = M_B^c$$

1) $\det(A - \lambda I) = 0$

2) $M_c^B = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = M$ $\qquad A = \overbrace{M \, D}\, M^{-1}$

2) $M_c^B = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = M$

$A = M D M$

3) $A^8 = \underbrace{M D^8 M^{-1}}$

$A^2 = \underbrace{M D \overbrace{M^{-1} M}^{I}} D M^{-1} =$

$A^2 = \underbrace{M D^2 M^{-1}}$

$A^3 = M D^3 M^{-1}$

Lay §5.7 page 333

Complex Eigenvalues

$X' = A X$ $\quad A = [a_{ij}] \quad a_{ij} \in \mathbb{R}$

$A = \overline{A}$ (conjugate of A)

Statement:

$\left[A v = \lambda v, \text{ with } \lambda \in \mathbb{C} \right] \Rightarrow$ [$\overline{\lambda}$ is also eigenvalue of A & $\overline{v}$ is also eigenvector of A associated to $\overline{\lambda}$]

Proof: $A v = \lambda v \Rightarrow \overline{A v} = \overline{\lambda v} \Rightarrow$

$\overline{A} . \overline{v} = \overline{\lambda} . \overline{v} \Rightarrow$

$\boxed{A \overline{v} = \overline{\lambda} \overline{v}}$

eigenvalue

corresp. eigenvector.

Goal. How to express the solution of $\underline{X' = A X}$ in terms of real functions.

Example 2x2 $\quad A_{2\times2} < \begin{matrix} \lambda \to v \\ \overline{\lambda} \to \overline{v} \end{matrix}$

$$\boxed{X = \underbrace{c_1 v e^{\lambda t}}_{c_1 X_1} + \underbrace{c_2 \bar{v} e^{\bar{\lambda} t}}_{c_2 X_2}}$$

Complex formulation.

$\lambda \in \mathbb{C} \Rightarrow \lambda = a + bi \qquad i = \sqrt{-1}$

$\bar{\lambda} = a - bi \qquad a = Re(\lambda) \quad b = Im(\lambda)$

$$v = \begin{pmatrix} c_1 + d_1 i \\ c_2 + d_2 i \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + i \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \qquad Re(v) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

$$Im(v) = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

$$e^{\lambda t} = e^{(a+bi)t} = e^{at + bti} = e^{at} \cdot e^{bti} =$$

$$e^{\lambda t} = e^{at}(\cos bt + i \sin bt)$$

$$c_1 X_1(t) = v \cdot e^{\lambda t} = [Re(v) + i\, Im(v)]\, e^{at} (\cos bt + i \sin bt)$$

$$= c_1 e^{at} \Big[(Re(v) \cos bt - Im(v) \sin bt) +$$

$$i\,(Re(v) \sin bt + Im(v) \cos bt) \Big]$$

$$c_1 X_1(t) = c_1 y_1(t) + c_1 i\, y_2(t)$$

$y_1(t) = e^{at}(Re\, v \cos bt - Im(v) \sin bt)$

$y_2(t) = e^{at}(Re\, v \sin bt + Im(v) \cos bt)$

$$c_2 X_2(t) = c_2 y_1(t) - c_2 i\, y_2(t)$$

$$X = c_1 y_1(t) + c_1 i\, y_2(t)$$
$$+$$
$$c_2 y_1(t) - c_2 i\, y_2(t)$$

$$X = (c_1+c_2)\, y_1(t) + (c_1 i - c_2 i)\, y_2(t)$$

So the formulation of the solution now in terms of real functions is given by

$$X(t) = \alpha\, y_1(t) + \beta\, y_2(t) \quad , \alpha, \beta \in \mathbb{R}$$

$$y_1(t) = e^{at}\left[\operatorname{Re}(v) \cos bt - \operatorname{Im}(v) \sin(bt) \right]$$

$$y_2(t) = e^{at}\left[\operatorname{Re}(v) \sin bt + \operatorname{Im}(v) \cos(bt) \right]$$

being $\lambda = a + bi$ & $v \equiv$ the corresponding eigenvector.

Example: Obtain λ, v such that $Av = \lambda v$ Lay page 314/315

$$A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$$

1) $A v = \lambda v \Leftrightarrow A v - \lambda I v = \vec{0} \Leftrightarrow (A - \lambda I)\vec{v} = \vec{0} \quad (a)$

To find non trivial solution for (a), $\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} 0.5-\lambda & -0.6 \\ 0.75 & 1.1-\lambda \end{pmatrix} = 0 \Rightarrow$$

$$(0.5-\lambda)(1.1-\lambda) + (0.6)\cdot(0.75) = 0$$

$$0{,}55 - 0.5\lambda - 1.1\lambda + \lambda^2 + 0{,}45 = 0$$

$$\lambda^2 - 1.6\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{1.6 \pm \sqrt{2.56 - 4.1.1}}{2}$$

$$\lambda_{1,2} = \frac{1.6 \pm \sqrt{-1.44}}{2}$$

$$\lambda_1 = \frac{1.6 + 1.2i}{2} = 0.8 + 0.6i$$

$$\lambda_2 = \frac{1.6 - 1.2i}{2} = 0.8 - 0.6i$$

$$(\lambda_2 = \overline{\lambda_1})$$

2) v_{λ_1}. Now Let's compute the associated eigenvalue.

$$\begin{pmatrix} 0.5-(0.8+0.6i) & -0.6 \\ 0.75 & 1.1-(0.8+0.6i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.3-0.6i & -0.6 \\ 0.75 & +0.3-0.6i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Obs: Row 1 & Row 2 are LD

So this allows us to avoid the Gaussian elimination step in a 2x2 system and decide for just one of the Rows of the system to compute its solution.

so choosing Row 2 we have:

$$0.75x + (0.3-0.6i)\,y = 0 \Rightarrow$$

$$0.75x = (-0.3+0.6i)\,y$$

$$x = \left(-\frac{0.3}{0.75} + \left(\frac{0.6}{0.75}i\right)\right)y = (-0.4+0.8i)\,y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = y\begin{pmatrix} -0.4+0.8i \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} -0.4+0.8i \\ 1 \end{pmatrix} \quad \alpha \in \mathbb{R}$$

Choosing $y = 5$, we have

$$\bar{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2+4i \\ 5 \end{pmatrix} \therefore \mathrm{Re}(v) = \begin{pmatrix} -2 \\ 5 \end{pmatrix};\ \mathrm{Im}(v) = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

According to what we proved: $\bar{v}$ is the eigenvector associated to $\bar{\lambda}$. So

$$\bar{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2-4i \\ 5 \end{pmatrix} \quad \mathrm{Re}\,\bar{v} = \begin{pmatrix} -2 \\ 5 \end{pmatrix};\ \mathrm{Im}\,\bar{v} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

Se pages 315 / 316 Lay for a nice geometric interpretation of the linear transform associated to this matrix.

Example 2. $A_{2\times 2}$ with $a_{ij} \in \mathbb{R}$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ compute $\underline{Av = \lambda v}$

Solution:

1) Determine eigenvalues λ. by $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{bmatrix} = 0 \Rightarrow (-\lambda)(-\lambda) - (-1)(1) = 0$$

$$\Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$$

2) $Av = \lambda v \iff (A - \lambda I)v = 0$, $\lambda = i$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Obs: we can do the gaussian elimination

OR we can remember that in a 2X2 Homogeneous s tem with infinito solutions, the rows have to be linearly Dependent.

So we choose Row 2.

$$x - iy = 0 \Longrightarrow x = iy$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} iy \\ y \end{pmatrix} = \overset{\alpha}{\cancel{y}} \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad \alpha \in \mathbb{C}.$$

if we choose $\alpha = -i$ we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i(-i) \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

So $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ or $v = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

2) Since $\lambda_2 = \overline{\lambda}$ $\quad \lambda_2 = -i$

& $v_{\lambda_2} = \overline{v}$ (conjugate)

$$\bar v = \begin{pmatrix} \bar i \\ \bar 1 \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \text{or} \quad \bar v = \begin{pmatrix} \bar 1 \\ \overline{-i} \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

So $$M_C^B = \begin{pmatrix} v & \bar v \end{pmatrix} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

OR

$$\begin{pmatrix} v & \bar v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \quad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \bar\lambda \end{pmatrix}$$

Tack för idag!