Föreläsning_7.2_alice

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Goal: Lay Chapter 7. - Diagonalization of Symmetric Matrices.

Lay pag 413

$A_{n\times n}$ is symmetric if $A = [a_{ij}] = [a_{ji}]$

$$A = A^T$$

Ex. $\begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix}$

- $A\vec{u} = \lambda \vec{u} \longrightarrow$ λ = eigenvalue of A
  u = eigenvector of A associated to λ.

$V_u = \text{Span}\{\vec{u}\}$ = eigenspace associated to u & λ.

$= \{\vec{w} = \alpha \vec{u} \,/\, \alpha \in \mathbb{R}\}$

$A\vec{v} = \beta \vec{v}$

Theorem 1 page 413 Lay

[If $A_{n\times n}$ is symmetric] [Then any 2 eigenvectors from different eigenspaces are orthogonal]

Proof

$A\vec{u} = \lambda \vec{u} \longrightarrow \vec{u}$

$A\vec{v} = \beta \vec{v} \longrightarrow \vec{v}$

? $\vec{u} \cdot \vec{v} = $?

$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad [u]^T = [\text{—}]$

$$\boxed{\lambda\vec{u} \cdot \vec{v}} = [\lambda u_1 \; \lambda u_2 \ldots \lambda u_n] \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{[v]} = [\lambda \vec{u}]^T [\vec{v}] \overset{\text{using}}{=} \underbrace{Au = \lambda u}_{*}$$

$$= [A\underbrace{\vec{u}}_{*}]^T [v] = [u^T] A^T [v] = [u^T] \underbrace{A[v]}_{} \underset{\substack{\circledast \\ Av = \beta v}}{=}$$

$$= [u^T][\beta v] = \beta [u]^T \cdot [v] = \beta \vec{u} \cdot \vec{v} \Rightarrow$$

$$\lambda \bar{u}\cdot\bar{v} = \beta \bar{u}\cdot\bar{v} \Rightarrow \lambda \bar{u}\cdot v - \beta \bar{u}\cdot\bar{v} = 0$$

$$\underbrace{(\lambda-\beta)}_{\neq}\bar{u}\cdot\bar{v} = 0 \Rightarrow \bar{u}\cdot\bar{v} = 0 \Leftrightarrow \boxed{\vec{u} \perp \vec{v}}$$

λ & β are distinct eigenvalues of A

Definition: $A_{n\times n}$ is Orthogonally diagonalizable

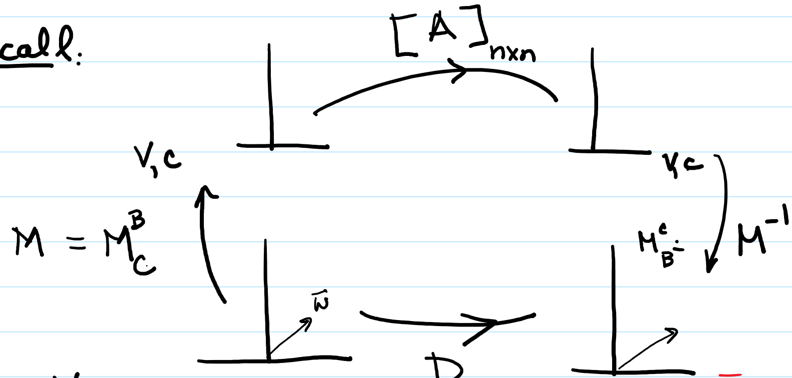
if there exist: 1) P = orthogonal matrix with eigenvectors ($P^{-1} = P^T$)

2) D = diagonal matrix $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

such that

$$\boxed{A = P D P^T = P D P^{-1}}$$

Recall:



c = standard basis

B = orthonormal basis of eigenvectors

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} - v_1 - \\ \vdots \\ - v_n - \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1\cdot v_1 & \cancel{v_1\cdot v_2} & \cdots & \cancel{v_1\cdot v_n} \end{bmatrix}$$

$$\underbrace{D \cancel{[w]_B}} = \underbrace{M^{-1} A M_C^B} \cancel{[\bar{w}]_B}$$

$$M^B D M^{-1} = M^B M^{-1} A M^B M^{-1}$$

$$\underbrace{}_{I}$$

$$M_c^B D M^{-1} = A$$

↓ Explanation Repetition

$$\boxed{M^{-1} = M^T}$$

$$A = M_c^B D M^{-1}$$

$\updownarrow$ since A is symmetric

$$\boxed{A_{n\times n} = P D P^T}$$

<u>Theorem 2</u>: page 414 §7.1

$$A_{n\times n} \text{ orthogonally diagonalizabe} \iff \boxed{A = A^T}$$

A is symmetric

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Theorem 3 The Spectral Theorem for Symmetric Matrices

<u>$A_{n\times n}$ symmetric</u>

a) A has n real eigen<u>values</u> (counting multipliaty)

b) $\dim V_\lambda$ = multiplicity of λ

c) $V_{\alpha} \perp V_{\beta}$ if $\alpha \neq \beta$ eigenvalues

V_α = eigenspace = Span { eigenvectors associated to α }

d) A is <u>orthogonally diagonalizable</u>

$$\boxed{A = P D P^T} \quad (*)$$

$$P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix} \qquad v_i \cdot v_j = 0 \quad i \neq j$$

$$v_i \cdot v_i = 1 \qquad \|v_i\| = 1$$

$$P^T = P^{-1}$$

Spectral Decomposition

$A_{n \times n}$ = symmetric $\qquad A = PDP^T$

$$P = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix} \qquad \underbrace{A u_i = \lambda_i u_i} \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$P^T = \begin{bmatrix} - & u_1 & - \\ - & u_2 & - \\ & \vdots & \\ - & u_n & - \end{bmatrix}$$

$$A = PDP^T = \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}} \begin{bmatrix} - & u_1 & - \\ - & u_2 & - \\ & \vdots & \\ - & u_n & - \end{bmatrix}$$

$$\begin{array}{c} \quad\quad 1^{st} \\ \begin{bmatrix} | & | & & | \\ \lambda_1 u_1 & \lambda_2 u_2 & \dots & \lambda_n u_n \\ | & | & & | \end{bmatrix} \end{array} \begin{bmatrix} - & u_1 & - \\ - & u_2 & - \\ & \vdots & \\ - & u_n & - \end{bmatrix}$$

$$\boxed{A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \dots + \lambda_n u_n u_n^T}$$

Spectral Decomposition of A

breaks up A into pieces determined by the spectrum $\{\lambda_1, \lambda_2 \dots \lambda_n\}$ of A.

each term $(\lambda_i u_i u_i^T)$ is a $n \times n$ matrix with rank 1

$$A\bar{x} = \underbrace{\lambda_1 u_1 u_1^T \bar{x}}_{\text{proj}_{v_1} \bar{x}} + \dots + \lambda_n u_n u_n^T \bar{x}$$

$$+ \dots + \text{proj}_{v_n} \bar{x}$$

Example: $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad \lambda_1 = 8$, $\lambda_2 = 3$

$\lambda_1 = 8 \Rightarrow \vec{v}_{\lambda_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow \vec{v}_{\lambda_1} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$ $\qquad \|\bar{v}\| = \sqrt{2^2+1^2} = \sqrt{5}$

$$\left\| \frac{\bar{v}}{\|v\|} \right\| = 1$$

$\lambda_2 = 3 \rightarrow \vec{v}_{\lambda_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \rightarrow \vec{v}_{\lambda_2} = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

$$\frac{\bar{v}}{\sqrt{5}} = \frac{1}{\sqrt{5}}(2,1) \Rightarrow$$

$$\left\| \frac{\bar{v}}{\sqrt{5}} \right\| = \left\| \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \right\| = \sqrt{\frac{4}{5} + \frac{1}{5}} = \sqrt{\frac{5}{5}} = 1.$$

$$\boxed{A = P D P^T}$$

$$A = \underbrace{\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -\frac{1}{\sqrt{5}} & 2/\sqrt{5} \end{bmatrix}}_{P^T}$$

The Spectral Decomposition is given by:

$$\begin{bmatrix} 8 \cdot \frac{2}{\sqrt{5}} & 3 \cdot \left(-\frac{1}{\sqrt{5}}\right) \\ 8 \cdot \frac{1}{\sqrt{5}} & 3 \cdot \left(\frac{2}{\sqrt{5}}\right) \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} =$$

$$= \boxed{\underset{\sim}{8} \underbrace{\begin{bmatrix} 2/\sqrt5 \\ 1/\sqrt5 \end{bmatrix}}_{\vec u_1} \underbrace{\begin{bmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{bmatrix}}_{\vec u_1^T} + \underset{\sim}{3} \underbrace{\begin{bmatrix} -1/\sqrt5 \\ 2/\sqrt5 \end{bmatrix}}_{\vec u_2} \underbrace{\begin{bmatrix} -\frac{1}{\sqrt5} & 2/\sqrt5 \end{bmatrix}}_{u_2^T}}$$

spectral decomposition.

To confirm:

$$\underset{\sim}{8} \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/2 \end{bmatrix} + \underset{\sim}{3} \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$$

$$\begin{bmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{bmatrix} + \begin{bmatrix} 3/5 & -6/5 \\ -6/5 & 12/5 \end{bmatrix} = \begin{bmatrix} \frac{35}{5} & \frac{10}{5} \\ \frac{10}{5} & \frac{20}{5} \end{bmatrix} = \underbrace{\begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}}_{A\,!!}$$

Now let's check the information about projections:

$$\underbrace{A^c_c \vec x}= 8 \underbrace{\begin{bmatrix} 2/\sqrt5 \\ 1/\sqrt5 \end{bmatrix} \begin{bmatrix} 2/\sqrt5 & 1/\sqrt5 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3 \begin{bmatrix} -1/\sqrt5 \\ 2/\sqrt5 \end{bmatrix} \begin{bmatrix} -1/\sqrt5 & 2/\sqrt5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$8 \begin{bmatrix} 2/\sqrt5\ 2/\sqrt5 & 2/\sqrt5\ 1/\sqrt5 \\ 1/\sqrt5\ 2/\sqrt5 & 1/\sqrt5\ 1/\sqrt5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$8 \begin{bmatrix} 2/\sqrt5 \left(2/\sqrt5 \cdot x_1 + 1/\sqrt5 \cdot x_2 \right) \\ 1/\sqrt5 \left(2/\sqrt5\, x_1 + 1/\sqrt5\, x_2 \right) \end{bmatrix}$$

$\vec u_1 \cdot \vec x$

$\vec u_1 \cdot \vec x$

$$\underbrace{8\ (\vec u_1 \cdot \vec x)\ \vec u_1}$$

$$8\ \operatorname{proj}_{\vec u_1} \vec x \qquad + 3\ \operatorname{proj}_{\vec u_2} \vec x$$

§6.5 Least-Squares Problems

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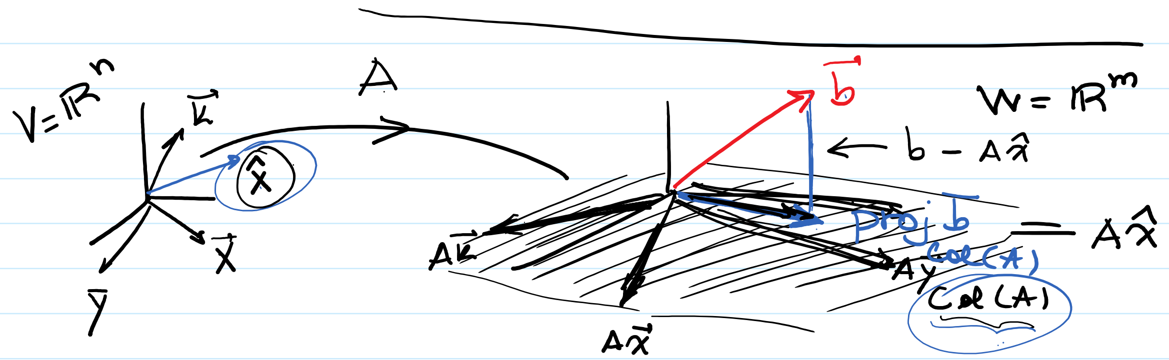
$$\boxed{\underbrace{A_{m\times n}}\ \underbrace{x_{n\times 1}} = \underbrace{b_{m\times 1}}}$$

$\mathbb{R}^n \to$ A $\to \vec b$ $W = \mathbb{R}^m$



If $A_{m\times n}$ matrix, $b \in \mathbb{R}^m$

A Least-squares solution of $Ax = b$

is a vector $\hat{x} \in \mathbb{R}^n$ such that

$$\underbrace{\| b - A\hat{x} \|} \le \underbrace{\| b - Ax \|} \qquad (= \text{ just when } b \in Col(A))$$

How to solve..

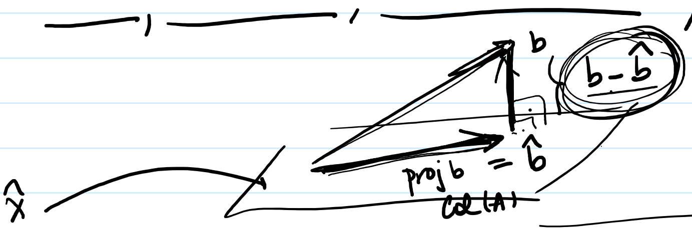
1) $\hat{b} = \text{proj}_{W = Col(A)}\, b$

$\hat{b} \in Col(A) \Rightarrow A x = \hat{b}$ now is consistent

2) $\exists\ \hat{x} \in \mathbb{R}^n$ such that $A\hat{x} = \hat{b}$

3) $\hat{x} \equiv$ an approximate solution for the original problem.

4) error $\approx \| b - \hat{b} \|^{2} = \sqrt{(b-\hat{b})\cdot(b-\hat{b})}$



$A\hat{x} = \hat{b}$

$$\boxed{b - \hat{b} \perp Col(A)}$$

$$\boxed{\vec{a}_i \cdot (b - \hat{b}) = 0} \quad i = 1 \dots n$$

$Col(A)$ $\vec{a}_i \cdot (b - A\hat{x}) = 0 \Leftrightarrow [\ a_i\ ]^T [\ b - A\hat{x}\ ] \Longrightarrow$

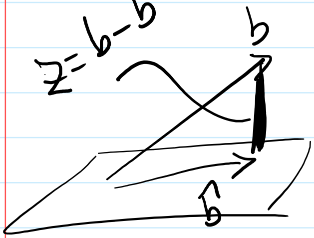
$z \perp col(A)$ $\quad \vec{a}_i \cdot (b - A\hat{x}) = 0 \iff \underset{\text{Row}}{[\; a_i \;]^T} \underset{\text{vector}}{\begin{bmatrix} b - A\hat{x} \end{bmatrix}} \implies$

$\|z\| \approx$ error

$$A^T \cdot \begin{bmatrix} b - A\hat{x} \end{bmatrix} = 0$$

$z = b - \hat{b}$

$$A^T \cdot b - A^T A \hat{x} = 0$$



$$\boxed{A^T A \hat{x} = A^T b}$$

This is called the NORMAL equation for $Ax = b$ (original system)

$A_{m \times n}$

$$A^T_{n \times m} A_{m \times n} = (A^T A)_{n \times n}$$

Ex1 from Lay (Chapter 6)

Find least squares solution for $\boxed{Ax = b}$

given by the exercise

$$A_{3\times 2} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \boxtimes \end{bmatrix} \quad ???$$

$\neq 0$

$$\Downarrow$$

$$\boxed{A^T A x = A^T b}$$

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{(17 \cdot 5 - 1)} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

page 387 Application to Linear Models

(Linear Regression)

given

x_i	y_i
x_1	y_1
$\vdots$	$\vdots$
x_n	y_n