2-nd order Linear ODE with constant coefficients

1) Homogeneous case : $ay'' + by' + cy = 0$; $a, b, c \in \mathbb{R}$
$y = y(t)$

Adams : § 3.7 page 206

Method of solution based on :
Hypothesis $y(t) = e^{rt}$, r to be determined

Lay : § 5.7 page 329

Method : $x'(t) = A\,x(t)$

Diagonalization of matrix A

Obs : • one special case of A represents the examples studied by Adams § 3.7

• the formulation presented by Lay § 5.7 is more general and enable us to study different systems of ODE , not only those specifically associated to $ay'' + by' + cy = 0$.

In any case, I would like to show that for equations $ay'' + by' + cy = 0$, both formulations (Adams & Lay) are equivalent and lead us to the same solution.
(As it should be!)

2) Non-Homogeneous case : $ay'' + by' + cy = f(t)$ $a, b, c \in \mathbb{R}$
$y = y(t)$

Solution $y(t) = y_H(t) + y_P(t)$

$y_H(t)$ = solution of the homogeneous part

$y_P(t)$ = particular solution depending on function $f(t)$.

Adams § 18.4 page 1017 general presentation
§ 18.5 1018 ODE with constant coeff. Higher order

Adams § 18.4 page 1017 general presentation
§ 18.5 page 1018 ODE with constant coeff Higher order (same hypothesis $y(t) = e^{rt}$)
§ 18.6 page 1025 Non-homogeneous case

Lay: The Non-homogeneous case is not presented on Lay considering the vector form.

Extra material is provided for those who have interest.

For the examination (Tentan MVE465 – 14/01/2021) we will Consider only the formulation presented by Adams for the Non-homogeneous case.

Example: Solve the Initial Value Problem (IVP)

$$\begin{cases} y'' - 3y' + 2y = \cos 3t \\ y(0) = 2 \\ y'(0) = -3 \end{cases}$$

1) Resolution according to Adams § 3.7 for the Homogeneous part & Adams § 18.6 for the Non-homogeneous part.

Solution: y_H = solution of the Homogeneous ODE $\boxed{y'' - 3y' + 2y = 0}$.

Hypothesis $\boxed{y(t) = e^{rt}}$, r to be determined.

$y = e^{rt}$
$y' = r e^{rt}$
$y'' = r^2 e^{rt}$

substituting $\Rightarrow$ on the Homog. Eq

$r^2 e^{rt} - 3r e^{rt} + 2e^{rt} = 0$

$(r^2 - 3r + 2) e^{rt} = 0 \Rightarrow \begin{cases} \cancel{e^{rt} = 0} \\ \text{OR} \\ (r^2 - 3r + 2) = 0 \end{cases}$

Characteristic equation: $\boxed{r^2 - 3r + 2 = 0}$

$r_{1,2} = \dfrac{3 \pm \sqrt{9 - 4.1.2}}{2.1} < \begin{matrix} \dfrac{3+1}{2} = \dfrac{4}{2} = 2 & r_1 = 2 \\ \dfrac{3-1}{2} = \dfrac{2}{2} = 1 & r_2 = 1 \end{matrix}$

So $\boxed{y_H(t) = c_1 e^{2t} + c_2 e^{t}}$ Homogeneous solution.

Now, to obtain y_p we have to analyse $f(t) = \cos(3t)$

So $y_p(t) = A\cos(3t) + B\sin(3t)$, A and B to be determined.

To substitute y_p into the equation, first we compute:

$$y_p = A\cos 3t + B\sin 3t$$
$$y_p' = -3A\sin 3t + 3B\cos 3t$$
$$y_p'' = -9A\cos 3t - 9B\sin 3t$$

The Non-Homogeneous EDO: $y'' - 3y' + 2y = \cos 3t = \underline{1}\cos 3t + \underline{0}\sin 3t$

$$\begin{array}{ll} y'' & -9A\cos 3t - 9B\sin 3t \\ -3y' & -3(-3A\sin 3t + 3B\cos 3t) \\ +2y & 2(A\cos 3t + B\sin 3t) \\ \hline = & \end{array}$$

$$1\cos 3t + 0\sin 3t \qquad (-9A - 9B + 2A)\cos 3t + (-9B + 9A + 2B)\sin 3t$$

$$\begin{cases} -7A - 9B = 1 \\ 9A - 7B = 0 \end{cases} \Rightarrow 9A = 7B \Rightarrow \boxed{A = \frac{7}{9}B}$$

$$-7\left(\frac{7}{9}\right)B - 9B = 1 \Rightarrow \left(-\frac{49}{9} - \frac{81}{9}\right)B = 1$$

$$\boxed{B = -\frac{9}{130}} \Rightarrow A = \frac{7}{\cancel{9}}\left(-\frac{\cancel{9}}{130}\right) = -\frac{7}{130}$$

$$\begin{array}{r} 81 \\ 49 \\ \hline 130 \end{array}$$

So the particular solution is given by:

$$\boxed{y_p = -\frac{7}{130}\cos(3t) - \frac{9}{130}\sin(3t)}$$

Now to obtain the solution for the IVP, we have to use the two information: $\begin{cases} y(0) = 2 \\ y'(0) = -3 \end{cases}$

$$\begin{cases} y(0) = 2 \\ y'(0) = -3 \end{cases}$$

$$y(t) = \underbrace{y_H(t) + y_p(t)}_{} = c_1 e^{2t} + c_2 e^{1t} - \frac{7}{130}\cos(3t) - \frac{9}{130}\sin(3t)$$

$$y'(t) = y_H'(t) + y_p'(t) = 2c_1 e^{2t} + c_2 e^{t} + \frac{21}{130}\sin(3t) - \frac{27}{130}\cos(3t)$$

$$\begin{cases} y(0) = c_1 + c_2 - \frac{7}{130} = 2 \\ y'(0) = 2c_1 + c_2 - \frac{27}{130} = -3 \end{cases}$$

$$\begin{cases} c_1 + c_2 = 2 + \frac{7}{130} = \frac{267}{130} \\ 2c_1 + c_2 = -3 + \frac{27}{130} = -\frac{363}{130} \end{cases}$$

$$\begin{array}{r} 130 \\ \times\ 3 \\ \hline 390 \end{array} \qquad \begin{array}{r} 390 \\ -\ 27 \\ \hline 363 \end{array}$$

$$\begin{cases} c_1 + c_2 = 267/130 \\ - \quad 2c_1 + c_2 = -363/130 \end{cases}$$

$$-c_1 + 0 = \frac{267}{130} = \frac{630}{130}$$

$$\begin{array}{r} 363 \\ +\ 267 \\ \hline 630 \end{array}$$

$$c_1 = -\frac{630}{130}$$

$$c_2 = \frac{267}{130} - c_1 = \frac{267}{130} + \frac{630}{130} = \frac{897}{130}$$

Thus the solution of the IVP is:

$$y(t) = -\frac{63}{13} e^{2t} + \frac{897}{130} e^{t} - \frac{7}{130}\cos(3t) - \frac{9}{130}\sin(3t) \qquad \square$$

Now, we just solve the Homogeneous part again via eigenvalues & eigenvectors to show the connection between the 2 formulations:

Recal: $\boxed{x'' - 3x' + 2x = 0}$ (Homogeneous Eq)
(Notation used in our class)

Solution:

1^{st}) Matricial formulation of the equation:

$$\underbrace{\begin{cases} x_1 = x \\ x_2 = x_1' = x' \end{cases}}_{(i)} \Rightarrow x_2' = x_1' = x'' \overset{\text{by the equation}}{\underset{(ii)}{=}} -2\underset{x_1}{x} + 3\underset{x_2}{x'}$$

Now,
$$\begin{cases} x_1' = 0x_1 + 1x_2 & (i) \\ x_2' = -2x_1 + 3x_2 & (ii) \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \boxed{X' = A_{2\times 2} X}$$

$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$ Let's try to diagonalize A.

2^{nd}) $\det(A - \lambda I) = 0$

$$\det\begin{pmatrix} 0-\lambda & 1 \\ -2 & 3-\lambda \end{pmatrix} = 0 \Rightarrow \boxed{(-\lambda)(3-\lambda) + 2 = 0} \Rightarrow$$

$$-3\lambda + \lambda^2 + 2 = 0 \Rightarrow \boxed{\lambda^2 - 3\lambda + 2 = 0}$$ Characteristic polynomial

Observe that this is the same equation we had before to compute r.

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9 - 4.1.2}}{2.1} \quad \begin{cases} \lambda_1 = \frac{3+1}{2} = \frac{4}{2} = 2 \\ \lambda_2 = \frac{3-1}{2} = \frac{2}{2} = 1 \end{cases}$$

λ_1 & λ_2 are eigenvalues of A.

3rd) Let's compute the eigenvectors associated to each λ_i.

$\lambda_1 = 2$ $$\begin{pmatrix} 0-2 & 1 \\ -2 & 3-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$-2x + 1y = 0 \Rightarrow 2x = y \Rightarrow x = \frac{1}{2}y$$

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y \\ y \end{pmatrix} = y\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$ is one possible eigenvector associated to $\lambda_1 = 2$.

$\lambda_2 = 1$ $$\begin{pmatrix} 0-1 & 1 \\ -2 & 3-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\div -2 \quad \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ +1 & -1 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

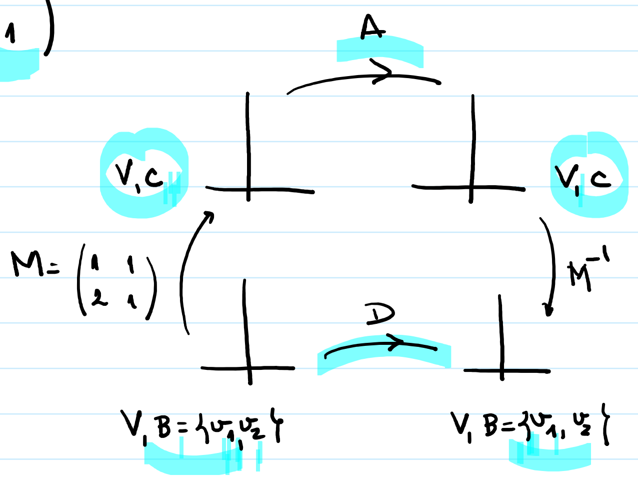
$$-x + y = 0 \Rightarrow x = y \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \alpha \in \mathbb{R}$$

So one eigenvector could be $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$B = \left\{ v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$ base of eigenvectors

$$M_C^B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

So $\boxed{X' = AX}$



$$M = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$ Matrix of Change of Basis from B to Canonical

' ' | ' of Basis from B to Canonical

$V_1\ B=\{v_1, v_2\}$ $\quad$ $V_1\ B=\{v_1, v_2\}$

$$[D]_B^B = M^{-1} A_c^c M_c^B$$

$$\boxed{x = M_c^B y}$$

y is the solution given on the Basis of Eigenvectors

$$\boxed{M^{-1} x = y}$$

So the change of variables is given by

$$x'(t) = A\, x(t)$$

$$M^{-1} x'(t) = M^{-1} A \underbrace{(M M^{-1})}_{I} x$$

$$\underbrace{P^{-1} x'(t)}_{} = \underbrace{(M^{-1} A M)}_{}(M^{-1} x)$$

$$y(t) = \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}}_{D} y(t)$$

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \Rightarrow$$

$$y_1'(t) = 2 y_1(t) \Rightarrow y_1(t) = c_1 e^{2t}$$

$$y_2'(t) = 1\ y_1(t) \Rightarrow y_2(t) = c_2 e^{1t}$$

So $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{1t} \end{pmatrix}$

$$x = M_c^B y \Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{1t} \end{pmatrix} =$$

$$x(t) = \begin{pmatrix} x_1(t) \\ \cdots \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ \cdot \end{pmatrix} + c_2 e^{1t} \begin{pmatrix} 1 \\ \cdot \end{pmatrix}$$

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{1t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{2t} + c_2 e^{1t} \\ 2c_1 e^{2t} + c_2 e^{t} \end{pmatrix} = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$$

$\boxed{x(t) = c_1 e^{2t} + c_2 e^{1t}}$ is exactly what we have obtained with the methodology given by Adams. !

The link for the webpage where Non-homogeneous ODE are treated by the matricial formulation is given directly on the Canvas web-page.