Gram-Schmidt Orthogonalization Process

Lay Page 372 §6.4

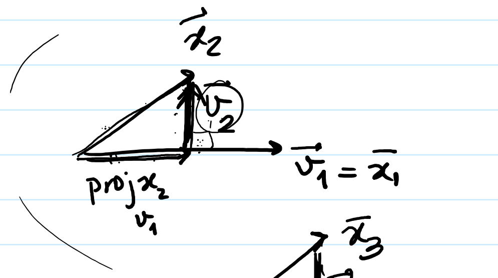
Theorem 11 page 373

Given a basis $B=\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ for a nonzero subspace $W \subseteq \mathbb{R}^n$

Define

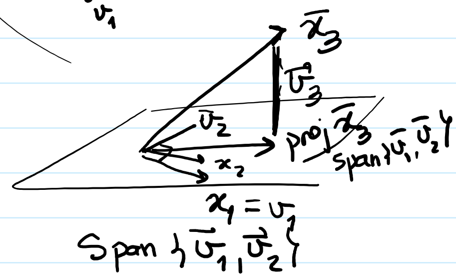
$$\vec{v}_1 \leftarrow \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \operatorname{proj}_{\vec{v}_1} \vec{x}_1 = \vec{x}_2 - \frac{\vec{x}_1 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$



$$\vec{v}_3 = \vec{x}_3 - \operatorname{proj}_{\operatorname{Span}\{\vec{v}_1, \vec{v}_2\}} \vec{x}_3 =$$

$$\vec{x}_3 - \left(\operatorname{proj}_{\vec{v}_1} \vec{x}_3 + \operatorname{proj}_{\vec{v}_2} \vec{x}_3 \right)$$



$$\vec{v}_4 = \vec{x}_4 - \operatorname{proj}_{\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}} \vec{x}_4 =$$

$$= \vec{x}_4 - \left(\operatorname{proj}_{\vec{v}_1} \vec{x}_4 + \operatorname{proj}_{\vec{v}_2} \vec{x}_4 + \operatorname{proj}_{\vec{v}_3} \vec{x}_4 \right)$$

$$\vec{v}_n = \vec{x}_n - \operatorname{proj}_{\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\}} \vec{x}_n$$

Then $\mathcal{O} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal basis of $W \subseteq \mathbb{R}^n$

& $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \operatorname{Span}\{\vec{x}_1, \dots, \vec{x}_k\}$ $1 \le k \le n$.

Example. Consider $B = \{(1,1,1), (0,2,1), (0,0,1)\}$ basis for $W \subseteq \mathbb{R}^3$.

- Obtain $\mathcal{O}$ orthogonal Basis via Gram-Schmidt having B as the starting point
- Obtain $\mathcal{O}'$ orthonormal Basis related to $\mathcal{O}$.

having O as the starting point

- Obtain $\mathcal{O}'$ orthonormal Basis related to $\mathcal{O}$.

Gram-Schmidt

$B = \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ $\quad \bar{x}_1 = (1,1,1) \quad \bar{x}_2 = (0,2,1) \quad \vec{x}_3 = (0,0,1)$

1) $\bar{v}_1 = \bar{x}_1 = (1,1,1)$

$$\bar{v}_2 = \bar{x}_2 - \mathrm{proj}_{\bar{v}_1} \bar{x}_2 = (0,2,1) - \frac{(0,2,1)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1)$$

2) $\vec{v}_2 = (0,2,1) - \frac{\cancel{3}}{\cancel{3}}(1,1,1) = (-1,1,0)$

$$\vec{v}_3 = \bar{x}_3 - \mathrm{proj}_{\mathrm{Span}\{\bar{v}_1,\bar{v}_2\}} \bar{x}_3 = \bar{x}_3 - \mathrm{proj}_{\bar{v}_1} \bar{x}_3 - \mathrm{proj}_{\bar{v}_2} \bar{x}_3$$

$$= (0,0,1) - \frac{(0,0,1)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) - \frac{(0,0,1)\cdot(-1,1,0)}{(-1,1,0)\cdot(-1,1,0)}(-1,1,0)$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - 0(-1,1,0)$$

$$\bar{v}_3 = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right)$$

So $\mathcal{O} = \Big\{ \underbrace{(1,1,1)}_{\bar{v}_1 = \bar{x}_1}, \underbrace{(-1,1,0)}_{\bar{v}_2}, \underbrace{\left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right)}_{\bar{v}_3} \Big\}$

Orthogonal Basis based on B.

Observe that

$\|\bar{v}_1\| = \sqrt{1+1+1} = \sqrt{3} \neq 1$ $\quad$ $\bar{v}_1$ is not an unitary vector.

$\|\bar{v}_2\| = \sqrt{1^2+1^2+0} = \sqrt{2} \neq 1$ $\quad$ $\bar{v}_2$ has size different from 1.

$\|\bar{v}_3\| = \sqrt{\left(\frac{1}{9}\right) + \frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$ $\quad$ $\bar{v}_3$ is not normalized.

So the orthonormal basis $\mathcal{O}'$ obtained throught $\mathcal{O}$ is

$$\mathcal{O}' = \left\{ \frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \frac{\vec{v}_3}{\|\vec{v}_3\|} \right\}$$

$$= \left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\sqrt{6}}, -\frac{1}{\cancel{3}} \frac{\cancel{3}}{\sqrt{6}}, \frac{2}{\cancel{3}} \frac{\cancel{3}}{\sqrt{6}}\right) \right\}$$

$$\mathcal{O}' = \left\{ \underbrace{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}_{\overline{w}_1}, \underbrace{\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}_{\overline{w}_2}, \underbrace{\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)}_{\overline{w}_3} \right\}$$

Obs.

$$\overline{w}_1 = \frac{\overline{v}_1}{\|\vec{v}_1\|} \qquad \|\vec{w}_1\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \sqrt{\frac{3}{3}} = 1$$

$$\overline{w}_2 = \frac{\overline{v}_2}{\|\vec{v}_2\|} \qquad \|\vec{w}_2\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

$$\overline{w}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} \qquad \|\vec{w}_3\| = \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}} = \sqrt{\frac{6}{6}} = 1$$

Actually $\overline{w} = \frac{\overline{u}}{\|\overline{u}\|}$ is allways normal since

$$\|\overline{w}\| = \left\| \frac{\overline{u}}{\underbrace{\|\overline{u}\|}_{cte}} \right\| = \frac{1}{\cancel{\|\overline{u}\|}} \cdot \cancel{\|\overline{u}\|} = 1. \quad \text{for all } \vec{u} \neq \vec{0}.$$

$(\forall \overline{u} \neq \vec{0})$

$\square$.