MVE 465 : Linjär Algebra och Analys fortsättning HT20

Course Overview : Look at the Program in Canvas.

Vecka 1 **Integration & Applications**

Adams Chapter 5 : sections 5.1; 5.2; 5.3; 5.4; 5.5; 5.6; 5.7

Chapter 2 : section 2.10

Chaper 6 : section 6.1

Connection with MVE460

1) $\lim_{h \to 0} f(x+h) = f(x)$

2) $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \Rightarrow$ Tangent line inclination
Geometric Interpretation

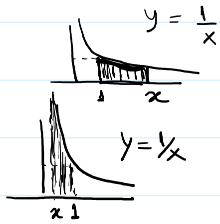
3) $y = f(x)$: $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$

$dy = \frac{dy}{dx} . dx = f'(x)\, dx$ Differential

This is our starting point for MVE 465

4) The Natural logarithm

Area $\left\{ \ln x = \begin{cases} A_x \ ; & x \geq 1 \\ -A_x \ ; & 0 < x < 1 \end{cases} \right.$



$$\int \frac{1}{x} dx = \underbrace{\ln|x| + C}_{F(x)}$$

Primitive function

Problem: Given a funktion f, is there another function F such that $\frac{dF(x)}{dx} = f(x)$?

Definition: Let f be differentiable on an interval $I \subseteq \mathbb{R}$.
A differentiable function F is called Primitive function of f if $\frac{d}{dx} F(x) = f(x)$.

$$\boxed{F'(x) = f(x), \ \forall x \in I}$$

Ex: $F(x) = \arctan(\sqrt{x})$ is a primitiv of $f(x) = \dfrac{1}{2\sqrt{x}\,(1+x)}$, since:

Solution:

$$\frac{dF(x)}{dx} = \frac{d}{dx}\left(\arctan(\sqrt{x})\right) = \frac{1}{(1+(\sqrt{x})^2)} \cdot \frac{1}{2}(x)^{\frac{1}{2}-1} = \frac{1}{(1+x)} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = f(x) \qquad \square$$

Property: If $F(x)$ is a primitiv of f, then $G(x) = F(x) + C$, $C \in \mathbb{R}$ is also a primitiv of f.

On the other hand, the constant C is the only indetermination that a primitiv of f can have.

Assuming that $G(x)$ is another primitiv of f

$G'(x) = f = F'(x)$ but this implies that [(*) see Tentan 1]

$G(x) = F(x) + C$

So $f'(x)$ is the derivative of $f(x)$

And $f'(x)dx$ is the differential of $f(x)$

Now $f(x)$ is the derivative of $F(x)$

And $f(x)dx$ is the differential of $F(x)$

And this is the background that allows

$$\int f(x)dx = F(x) + C$$

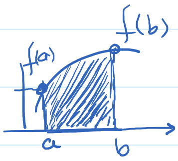
(*) Again a conexion to limit Theorems

$$\int f(x)\,dx = \underbrace{F(x) + C}_{\text{primitiv function}} \iff \underbrace{\int_a^b f(x)\,dx}_{\text{Definite Integral}} \overset{(*)}{\underset{(*)}{=}} F(b) - F(a)$$

Definite Integral

$\Updownarrow$

Related to the Area under a curve



So before coming to the text Lecture_1.1_integrals which deals with Area

I would like to mention a bit more about primitiv functions:

A list to Recall - Examples:

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1) $\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C$; $\alpha \in \mathbb{R}$, $\alpha \neq -1$, $C \in \mathbb{R}$

2) $\int \frac{1}{x}\,dx = \ln|x| + C$ $\quad x \neq 0$

3) $\int e^{x}\,dx = e^{x} + C$

4) $\int \cos x\,dx = \sin x + C$

5) $\int \sin x\,dx = -\cos x + C$

6) $\int \frac{1}{\cos^2 x}\,dx = \tan x + C$ $\quad x \neq \pm\pi/2, \pm\frac{3\pi}{2}, \dots$

7) $\int \frac{1}{\sin^2 x}\,dx = -\cot x + C$ $\quad x \neq 0, \pm\pi, \pm 2\pi, \dots$

8) $\int \frac{1}{1+x^2}\,dx = \arctan x + C$ $\quad (x \neq \pm 1)$

9) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$ $\quad (x \neq 1, -1)$

10) $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + C$ $\quad (x \neq \pm 1)$

Ex: Verify that $\int \frac{1}{\sqrt{x^2+\alpha}}\,dx = \ln\left|x + \sqrt{x^2+\alpha}\right| + C$ $\quad \alpha > 0$, $C \in \mathbb{R}$, $x^2+\alpha \neq 0$.

Solution:

given a candidate to the Primitive $F(x) = \ln|x+\sqrt{x^2+\alpha}| + C$,
we only have to compute $\frac{d}{dx}F(x)$ & compare with $f(x)$.

$$\frac{d}{dx}\left(\ln\left|x+\sqrt{x^2+\alpha}\right|+C\right) = \frac{d}{dx}\left(\ln\left|x+\sqrt{x^2+\alpha}\right|\right) \overset{\text{Chain Rule.}}{=}$$

$$= \frac{1}{(x+\sqrt{x^2+\alpha})}\cdot\frac{d}{dx}\left(x+\sqrt{x^2+\alpha}\right) = \frac{1}{(x+\sqrt{x^2+\alpha})}\left(1+\frac{1}{\cancel{2}}(x^2+\alpha)^{\frac{1}{2}-1}\cdot\cancel{2}x\right) =$$

$$= \frac{1}{(x+\sqrt{x^2+\alpha})}\left(1+\frac{x}{\sqrt{x^2+\alpha}}\right) =$$

$$= \frac{1}{\cancel{(x+\sqrt{x^2+\alpha})}}\cdot\left(\frac{\cancel{\sqrt{x^2+\alpha}+x}}{\sqrt{x^2+\alpha}}\right) = \frac{1}{\sqrt{x^2+\alpha}} \quad \square$$

Some Important Rules to compute Indefinite Integrals ≡ Primitive functions.

1. $\int \frac{1}{f(x)}\cdot f'(x)\,dx = \ln|f(x)| + C$ (Chain Rule)

2. $\int f(x)^{\alpha}\cdot f'(x)\,dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + C \qquad \alpha\neq -1.$

3. $\int \alpha f(x)\,dx = \alpha\int f(x)\,dx, \quad \alpha\in\mathbb{R}.$

4) $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

Example: Compute ≡ Find the primitive

$$\int \frac{1+2x+3x^2}{x^3}\,dx$$

Solution:

$$\int \frac{1+2x+3x^2}{x^3}\,dx = \int\frac{1}{x^3}\,dx + \int\frac{2x}{x^3}\,dx + \int\frac{3}{x}\,dx =$$

$$= \underbrace{\int x^{-3}\,dx}_{} + 2\int x^{-2}\,dx + 3\int \frac{1}{x}\,dx$$

$$-\frac{1}{2}x^{-2} + \underbrace{c_1} + 2\left[x^{-1} + \underbrace{c_2}\right] + 3\left(\ln|x| + \underbrace{c_3}\right) =$$

$$-\frac{1}{2}x^{-2} + 2x^{-1} + 3\ln|x| + C, \quad C = (c_1 + 2c_2 + 3c_3) \in \mathbb{R}.$$

Ex 2 Compute

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{\cos x}(-\sin x)\,dx =$$

$--- = +$

$$= -\int \frac{1}{f(x)} \cdot f'(x)\,dx = -\ln|\cos x| + C, \quad C \in \mathbb{R}.$$

with $f(x) = \cos x$

$\square$

Now we study the aspects related to the geometric application of the definite integral $\equiv$ the area under a curve $y = f(x)$.