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Page 313 The Fundamental Theorem of Calculus

$$\left[F(x) = \int_a^x f(t)\,dt, \quad a \le x \le b \right] \Rightarrow \left[F \text{ is differentiable \& } F'(x) = f(x), \quad a < x < b \right]$$

Exercises:

1) Compute $F'(x)$ if $F(x) = \int_0^x e^{-t^2}\,dt$

Solution

$$F(x) = \int_0^x e^{-t^2}\,dt \quad \text{is a primitive of} \quad e^{-x^2}$$

So, by the Fund. Theorem of Calculus

$$F'(x) = \frac{dF}{dx} = e^{-x^2}$$

Obs ($F(x)$ can not be computed by elementary functions)

Exemple 2: Compute $F'(t)$, if $F(t) = \int_{\sqrt{t}}^{2} e^{-u^2}\,du$

Example 2: Compute $F'(t)$, if $F(t) = \int_0^{2\sqrt{t}} e^{-u^2}\,du$

Solution: By the Fundamental Theorem of Calculus, we have:

$$S(x) = \int_0^x e^{-u^2}\,du \implies S(2\sqrt{t}) = F(t)$$

$$x = 2\sqrt{t} \implies \frac{dx}{dt} = 2\cdot\frac{1}{2}(t)^{-1/2} = \frac{1}{\sqrt{t}}$$

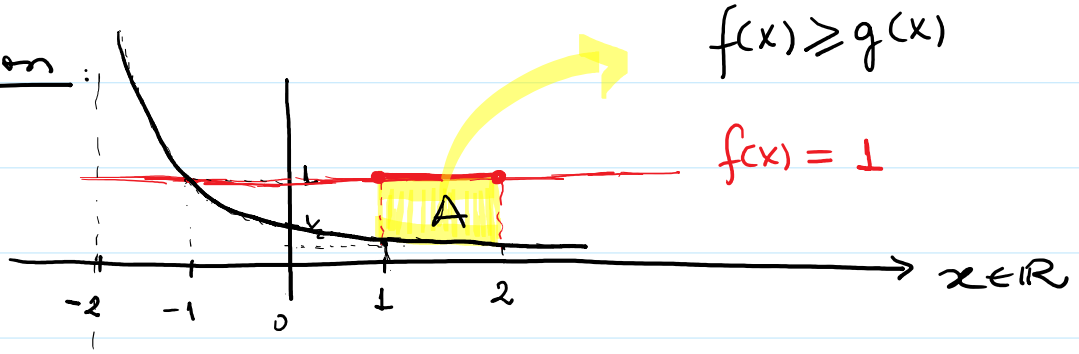
So by the Chain Rule we have

$$\frac{dF}{dt} = \underbrace{\frac{dS}{dx}}_{\text{FTC}} \cdot \frac{dx}{dt} = e^{-x^2}\cdot\frac{1}{\sqrt{t}} = e^{-(2\sqrt{t})^2}\frac{1}{\sqrt{t}}$$

$$= e^{-4t}\cdot\frac{1}{\sqrt{t}} \qquad \square$$

Example 3. Compute the Area between the two curves $y = f(x) = 1$, $x \in \mathbb{R}$ & $y = g(x) = \frac{1}{x+2}$ $x > -2$. in the interval $I = [1,2]$

Solution:



We want to compute

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$$\int_1^2 f(x)-g(x)\,dx = \underbrace{\int_1^2 \left(1-\frac{1}{x+2}\right)dx}_{*} \quad \left(= \int_1^2 \frac{x+2-1}{x+2}\,dx\right)$$

$$\int_1^2 1\,dx - \int_1^2 \frac{1}{(x+2)}\,dx \qquad x+2>0 \text{ when } x\in[1,2]$$

$$x\Big|_{x=1}^{x=2} - \ln|x+2|\Big|_1^2$$

$$(2-1) - (\ln|4| - \ln|3|) =$$

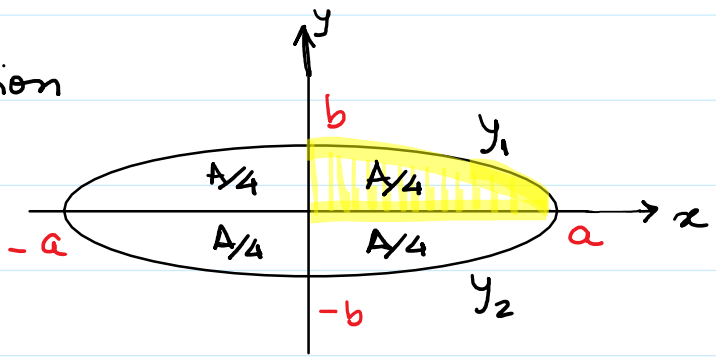
$$1 - (\ln(4) - \ln(3)) =$$

$$1 - \ln\left(\tfrac{4}{3}\right)$$

□

Example 4 . Compute the area of an ellipse with equation given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution



The ellipse is a plane curve which is symmetric. This symmetry can help to simplify the computation of the area A.

$$A_0 = 4\cdot\left(\frac{A}{4}\right)$$

So $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2} = 1-\frac{x^2}{a^2} \Rightarrow y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$

$$y = \pm\sqrt{b^2(1-x^2)} = \pm b\sqrt{1-x^2} \quad (*)$$

$$y_1 = +\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)} = +b\sqrt{1-\frac{x^2}{a^2}} \quad (*)$$

$$y_2 = -\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)} = -b\sqrt{1-\frac{x^2}{a^2}}$$

$$\frac{A}{4} = \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx \Rightarrow$$

$$\frac{A}{4b} = \int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx$$

Change of variable: $x = a\sin t \quad 0 \le t \le \pi/2$

$t=0 \Rightarrow x = a\sin(0) = 0$

$t=\pi/2 \Rightarrow x = a\sin(\pi/2) = a$

$$\frac{dx}{dt} = a\frac{d\sin t}{dt} \Rightarrow$$

$$\frac{dx}{dt} = a\cos t \Rightarrow \boxed{dx = a\cos t\,.\,dt}$$

So $$\frac{A}{4b} = \int_0^{\pi/2} \sqrt{1-\left(\frac{a\sin t}{a}\right)^2}\,.\,a\cos t\,dt =$$

$$\frac{A}{4ab} = \int_0^{\pi/2} \sqrt{1-\sin^2 t}\,\cos t\,dt \quad (*)$$

Using The trigonometric identities:

$$\frac{A}{4ab} = \int_0^{\pi/2} \sqrt{\cos^2 t}\,.\cos t\,dt =$$

$$\int_0^{\pi/2} |\cos t|\cos t\,dt$$

Since $0 \le t \le \pi/2$, $\cos t \ge 0$.

$$\int_0^{\pi/2} \cos^2 t\,dt = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2t)\,dt. \quad \begin{cases} y = 2t \\ dy = 2dt \end{cases}$$

$$\int_0^{\pi/2} \cos^2 t\, dt = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2t)\, dt. \quad \left\{ \begin{matrix} 0 \\ dy = 2dt \end{matrix} \right.$$

$$\frac{1}{2}\left[[t]_0^{\pi/2} - \left[\frac{1}{2}\sin(2t)\right]_0^{\pi/2} \right] =$$

$$\frac{1}{2}\left[(\pi/2 - 0) - \frac{1}{2}(\sin\pi - \sin 0)\right] =$$

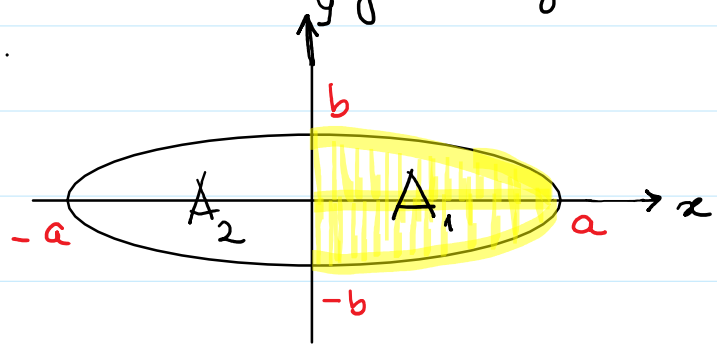
$$\frac{\pi}{4}$$

So $\dfrac{A}{4ab} = \dfrac{\pi}{4} \Rightarrow A = \dfrac{4ab\cdot\pi}{4} = \boxed{\pi \cdot ab}$.

Exercise 5 ≡ Related to Ex 4.

Compute A = area of Ellipse

Using Horizontal Integration.



$$A = A_1 + A_2 = 2A_1$$

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$x^2 = a^2\left(1 - \frac{y^2}{b^2}\right) \qquad x = +\sqrt{a^2\left(1-\frac{y^2}{b^2}\right)} \; ; \; -b \le y \le b.$$

$$x = a\sqrt{1 - \frac{y^2}{b^2}} \qquad -b \le y \le b.$$

$$A = 2\int_{-b}^{b} a\sqrt{1-\frac{y^2}{b^2}}\, dy = 2a\int_{-b}^{b}\sqrt{1-\frac{y^2}{b^2}}\, dy$$

Change of variable $y = b\sin t$ $\quad -\pi/2 \le t \le \pi/2$

$t = -\pi/2 \Rightarrow y = b.(-1) = -b$

$t = \pi/2 \Rightarrow y = b.(1) = b$

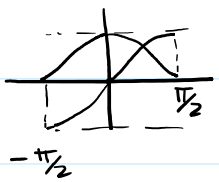
$$\frac{dy}{dt} = b\frac{d}{dt}\sin t = b\cos t$$

$$dy = b\cos t \cdot dt$$

$$\frac{A}{2a} = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \left(\frac{b\sin t}{b}\right)^2}\, b\cos t\, dt$$

$$\frac{A}{2ab} = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2 t}\, \cos t\, dt \overset{\text{Trigonometric identities}}{=}$$

$$\frac{A}{2ab} = \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2 t}\, \cos t\, dt \overset{*}{=}$$



$\cos t > 0, \ -\pi/2 < t < \pi/2$

$$\frac{A}{2ab} = \int_{-\pi/2}^{\pi/2} \cos^2 t\, dt =$$

$$\frac{A}{2ab} = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1 + \cos 2t\right) dt =$$

$$\frac{A}{ab} = \int_{-\pi/2}^{\pi/2} 1\, dt + \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos 2t \cdot 2\, dt$$

$$= \left[t\right]_{-\pi/2}^{\pi/2} + \frac{1}{2}\left[\sin 2t\right]_{-\pi/2}^{\pi/2}$$

$$\frac{A}{ab} = \pi/2 + \pi/2 + 0$$

$$\boxed{A = ab.\pi}$$

So both ways are equivalent. $\square$

$\boxed{A = ab.\pi}$ equivalent $\square$

Volume.