Föreläsning_2.1

den 9 november 2020 07:14

Mål

- Integration by substitution
- Volume, Area
- Length Element
- Area of revolution surfaces.

Recall - Integration by Part page 335

$$\left[\int u\,dV = UV - \int V\,dU \right]$$

Ex page 336: Compute: $\int \sin^{-1} x\,dx = \int 1 . \sin^{-1} x\,dx$

Solution:

$dV = 1 \Rightarrow V = x$

$U = \sin^{-1} x \Rightarrow dU = \dfrac{1}{\sqrt{1-x^2}}$

So $\int 1 . \sin^{-1} x\,dx = (\sin^{-1} x.)\,x - \underbrace{\int x . \dfrac{1}{\sqrt{1-x^2}}\,dx}_{\circledast \text{ To solve this}}$

$\circledast$ $\left(-\frac{1}{2}\right) \int \dfrac{1}{\sqrt{1-x^2}} \underbrace{(-2) . x\,dx} = -\dfrac{1}{2} \int \dfrac{1}{\sqrt{u}}\,du = -\dfrac{1}{2} \int u^{-1/2}\,du =$

$u = 1 - x^2$

$du = -2x\,dx$

$-\dfrac{1}{2}\left[\dfrac{u^{1/2}}{1/2} \right] = -u^{1/2} = -\sqrt{1-x^2}$

Back to the original problem

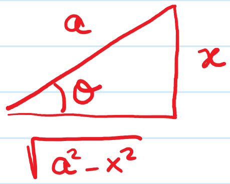
$$\int 1 . \sin^{-1} x\,dx = x . \sin^{-1} x - \left(-\sqrt{1-x^2}\right) + C$$

$$= x . \sin^{-1} x + \sqrt{1-x^2} + C$$

More about Substitutions page 349

- Trigonometric functions
- Inverse sine substitution

$\Rightarrow \cos\theta = \dfrac{\sqrt{a^2-x^2}}{a} \Rightarrow a\cos\theta = \sqrt{a^2-x^2}$



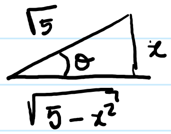
$$\Rightarrow \quad \cos\theta = \frac{\sqrt{a^2-x^2}}{a} \Rightarrow a\cos\theta = \sqrt{a^2-x^2}$$

$$\sin\theta = \frac{x}{a} \Rightarrow a\sin\theta = x$$

$$\tan\theta = \frac{x}{\sqrt{a^2-x^2}} \Rightarrow x = \tan\theta\sqrt{a^2-x^2}$$

Ex 1 page 350

$$\int \frac{1}{(5-x^2)^{3/2}}\,dx = \int \frac{1}{\left(\sqrt{(\sqrt{5})^2 - x^2}\right)^3}\,dx$$

⊛ 

$a = \sqrt{5}$

$$\sin\theta = \frac{x}{\sqrt{5}} \quad \begin{cases} x = \sqrt{5}\,.\sin\theta \\ dx = \sqrt{5}\cos\theta \end{cases}$$

Now. substitute this in the integrand

$$\int \frac{1}{\left(\sqrt{5 - 5\sin^2\theta}\right)^3}\sqrt{5}\cos\theta\,d\theta = \int \frac{1}{(\sqrt{5})^3\left(\sqrt{1-\sin^2\theta}\right)^3}\cancel{\sqrt{5}}\cos\theta\,d\theta$$

$$\int \frac{1}{(\sqrt{5})^2}\frac{1}{\left(\sqrt{\cos^2\theta}\right)^3}\cos\theta\,d\theta = \frac{1}{5}\int \frac{1}{\cos^3\theta}\cancel{\cos\theta}\,d\theta =$$

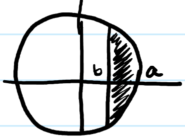
$$\frac{1}{5}\int \frac{1}{\cos^2\theta}\,d\theta = \frac{1}{5}\int \sec^2\theta\,d\theta \overset{⊛}{=} \underbrace{\frac{1}{5}\tan\theta + C}$$

⊛ $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{5-x^2}}$ thus $= \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + C$

Inverse sine Substitution page 350

Ex 2 [Find the area of the circular segment shaded in Figure 6.3

$y = \sqrt{a^2-x^2}$

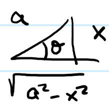


$$A = A_1 + A_2 = 2A_1$$

Solution: $A = 2\int_b^a \sqrt{a^2-x^2}\,dx$

$$\begin{cases} \text{substitution} \\ x = a\sin\theta \\ dx = a\cos\theta\,d\theta \end{cases}$$

$$A = 2\int_b^a \sqrt{a^2 - a^2\sin^2\theta}\;a\cos\theta\,d\theta =$$



$\sin\theta = \frac{x}{a}$

$\theta = \sin^{-1}\left(\frac{x}{a}\right)$

$$A = 2\int_b \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta =$$

$\sqrt{a^2-x^2}$ $\qquad \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$$2\int_b^a a^2\sqrt{\underbrace{1-\sin^2\theta}}\;\cos\theta\, d\theta = 2a^2\int \sqrt{\cos^2\theta}\;\cos\theta\, d\theta$$

$$= a^2\int_b^a 2\cos^2\theta\, d\theta \qquad \text{page } 325 \quad §5.6 \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= a^2\Big[\theta + \sin\theta\cos\theta\Big]_b^a$$

$$= a^2\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\,\frac{\sqrt{a^2-x^2}}{a}\right]_b^a$$

$$= a^2\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a^2}\sqrt{a^2-x^2}\right]_b^a$$

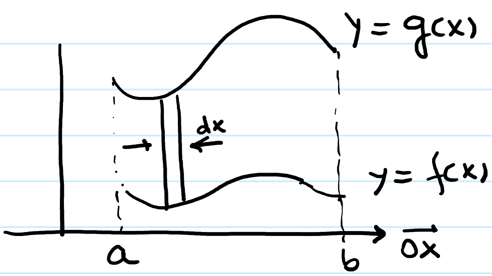
$$a^2\left[\left(\underbrace{\sin^{-1}\left(\frac{a}{a}\right)}_{\pi/2} + \frac{a}{a^2}\cancel{\sqrt{a^2-a^2}}\right) - \sin^{-1}\left(\frac{b}{a}\right) - \frac{b}{a^2}\sqrt{a^2-b^2}\right] =$$

$$\boxed{a^2\,\pi/2 - a^2\sin^{-1}\left(\frac{b}{a}\right) - b\sqrt{a^2-b^2}} \text{ square units } \square$$

Volume page 400

Volume of Solids of Revolution

(A) Region R



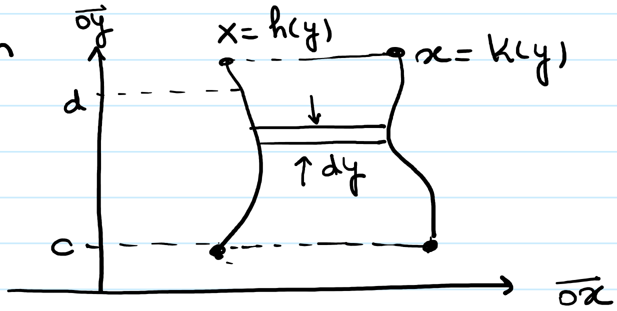
(*) Plane Slices $\therefore$ Rotation about $\vec{OX}$

$$V = \pi\int_a^b \left(g^2(x) - f^2(x)\right) dx$$

(*) Cylindrical Shells $\therefore$ Rotation about $\vec{OY}$

$$V = 2\pi \int_a^b x \cdot (g(x) - f(x))\, dx$$

(B) Region



★ Cylindrical Shell ∴ Rotation about $\overrightarrow{ox}$

$$V = 2\pi \int_c^d y\, (k(y) - h(y))\, dy$$

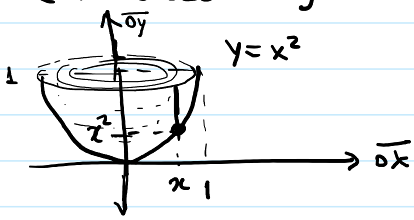
★ Plane Slices ∴ Rotation about $\overrightarrow{oy}$

$$V = \pi \int_c^d \left((k(y))^2 - (h(y))^2 \right) dy$$

Ex 7 page 399

[Find the volume of the bowl obtained revolving the parabolic arc $y = x^2$, $0 \leq x \leq 1$ about $\overrightarrow{oy}$.

Solution:



Rotation $\overrightarrow{oy}$ $\Rightarrow$ Cylindrical Shell
$y = f(x)$

$$V = 2\pi \int_0^1 x(x^2)\, dx \equiv \text{exterior part of the bowl}$$
(under the cury $y = x^2$)

(*) So $V = 2\pi \int_0^1 x\,(1 - x^2)\, dx \equiv$ Interior part of the bowl.

$$V = 2\pi \int_0^1 x - x^3 \, dx =$$

$$2\pi \left[\frac{x^2}{2} \Big|_0^1 - \left[\frac{x^4}{4} \right]_0^1 \right] =$$

$$2\pi \left[\tfrac{1}{2} - 0 - \tfrac{1}{4} \right] = 2\pi \cdot \tfrac{1}{4} = \pi/2 \text{ cubic units}$$

$\square$