§6.2 page 340 Rational Functions
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§6.5 page 363 Improper Integrals

$\int_a^{+\infty} f(x)dx$, $\int_{-\infty}^{+\infty} f(x)dx$, $\int_{-\infty}^{b} f(x)dx$

Integraler av rationella funktioner. Inversa variabelbyte. En samling av exempel från Adams som presenteras på föreläsningen den 2018.11.11 + 2020/11/12

Rationell funktion är funktion på formen

$$\boxed{R(x) = \frac{P(x)}{Q(x)}}$$

där P och Q är två polynom.

$\Rightarrow$ • $\partial P > \partial Q$ a) $P(x) \,|\, Q(x)$, $s(x)$, $r(x)$

$$\frac{P(x)}{Q(x)} = \frac{Q(x) \cdot s(x)}{Q(x)} + \frac{r(x)}{Q(x)}$$

• $\partial P < \partial Q$
$\partial r(x) < \partial Q$ $\Rightarrow$ factorization

EXAMPLE 1 Evaluate $\int \frac{x^3+3x^2}{x^2+1}\,dx.$

Solution The numerator has degree 3 and the denominator has degree 2 so we need to divide. We use long division:

$$\frac{P(x)}{Q(x)} = s(x) + \frac{r(x)}{Q(x)}$$

$$\begin{array}{r|l} & x + 3 \\ \hline x^2+1 & x^3 + 3x^2 \\ & x^3 \quad\quad + x \\ \hline & 3x^2 - x \\ & 3x^2 \quad + 3 \\ \hline & -x - 3 \end{array}$$

$$\boxed{\frac{x^3+3x^2}{x^2+1} = x+3 - \frac{x+3}{x^2+1}.}$$ Important

Thus,

$$\int \frac{x^3+3x^2}{x^2+1}\,dx = \int (x+3)\,dx - \int \frac{x}{x^2+1}\,dx - 3\int \frac{dx}{x^2+1}$$
$$= \frac{1}{2}x^2 + 3x - \frac{1}{2}\ln(x^2+1) - 3\tan^{-1}x + C.$$

$\int \frac{x}{x^2+1}dx$ $\left\{ \begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array} \right.$ =

$\int \frac{1}{u}du = \ln|u| + c = \ln|x^2+1| = \ln(x^2+1)$ since $(x^2+1)>0$

Huvudproblemet är att beräkna integral av rationell funktion $\frac{P(x)}{Q(x)}$ där P's grad är mindre än Q's grad. $\boxed{\partial P < \partial Q}$

Några elementära obestämda integraler of rationella funktioner

Tabellintegraler:

$\frac{P(x)}{Q(x)}$ $\partial P = 0$, $\partial Q = 1$ $\longrightarrow$ $\int \frac{c}{ax+b}dx = \frac{1}{a}c\ln\left|\frac{1}{a}(b+ax)\right| + C$

$\left|\frac{1}{a}(b+ax)\right| = \frac{1}{a}(b+ax)$ if $\frac{1}{a}(b+ax)>0$ $\forall x \in D_f$.

$\frac{P(x)}{Q(x)}$ $\partial P = 0$, $\partial Q = 2$ $\longrightarrow$ $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ $\leftarrow$ typo

$\int \frac{x}{x^2+a^2}dx = \frac{1}{2}\ln(x^2+a^2) + c$ since $x^2+a^2>0$, $\forall x \in D_f$.

Case $\frac{P(x)}{Q(x)}$ $\partial P = 0$ $\therefore P \equiv$ constant
$\partial Q = 2$, 2 real roots $\neq$.

$Q(x) = x^2+bx+c = (x-\alpha_1)(x-\alpha_2)$

Exempel. **Primitiv funktion av en enkel rationell funktion med nämnaren som har två olika reella rötter.**

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+C$$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} \overset{\text{New}}{=} \frac{A}{(x-a)} + \frac{B}{(x+a)} = \frac{Aa-Ba+Ax+Bx}{(x-a)(x+a)}$$

$$\frac{A}{(x-a)} + \frac{B}{(x+a)} = \frac{A(x+a)+B(x-a)}{(x-a)(x+a)}$$

$$\begin{cases} A+B &= 0 \\ Aa-Ba &= 1 \\ A &= \frac{1}{2a}; B = -\frac{1}{2a} \end{cases}$$

$$\int \frac{1}{x^2-a^2}dx = \int \frac{1}{2a(x-a)}dx - \int \frac{1}{2a(a+x)}dx =$$

$$\frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+C$$

properties of ln.

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$R(x) = \frac{P(x)}{Q(x)}$. Partial Fractions

$Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ $\partial Q = n$ = degree of Q.
$a_i \neq a_j$ $1 \leq i, j \leq n$. (distinct)

If $\partial P < \partial Q$, then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \frac{A_3}{(x-a_3)} + \dots + \frac{A_n}{(x-a_n)}$$

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$P(x) = x+4$ $\partial P = 1$
$Q(x) = x^2-5x+6$ $\partial Q = 2$

Exempel 3.

EXAMPLE 3 Evaluate $\displaystyle\int \frac{(x+4)}{x^2-5x+6}\,dx$.

Solution The partial fraction decomposition takes the form

$$\frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}. \quad (*)$$

We calculate A and B by both of the methods suggested above.

METHOD I. Add the partial fractions

$$\frac{1x+4}{x^2-5x+6} \overset{(*)}{=} \frac{Ax-3A+Bx-2B}{(x-2)(x-3)},$$

To compute A, B by solving a system, obtained by comparing both sides of $(*)$

$$B = \left.\frac{(x+4)}{(x-2)}\right|_{x=3}$$

and equate the coefficient of x and the constant terms in the numerators on both sides to obtain

$$\left.\frac{\cancel{(x-3)}\,x+4}{(x-2)\cancel{(x-3)}}\right|_{x=3} = \left.\frac{A\overset{=0}{\cancel{(x-3)}}}{(x-2)}\right|_{x=3} + \left.\frac{B\,\cancel{(x-3)}}{\cancel{(x-3)}}\right|_{x=3}$$

$A+B=1$ and $-3A-2B=4$

and equate the coefficient of x and the constant terms in the numerators on both sides to obtain

$$A + B = 1 \qquad \text{and} \qquad -3A - 2B = 4.$$

Solve these equations to get $A = -6$ and $B = 7$.

METHOD II. To find A, cancel $x - 2$ from the denominator of the expression $P(x)/Q(x)$ and evaluate the result at $x = 2$. Obtain B similarly.

$$A = \left.\frac{x+4}{x-3}\right|_{x=2} = -6 \qquad \text{and} \qquad B = \left.\frac{x+4}{x-2}\right|_{x=3} = 7.$$

In either case we have

$$\int \frac{(x+4)}{x^2 - 5x + 6}\,dx = -6\int \frac{1}{x-2}\,dx + 7\int \frac{1}{x-3}\,dx$$
$$= -6\ln|x-2| + 7\ln|x-3| + C.$$

$$\frac{\cancel{(x-3)}\,x+4}{(x-2)\cancel{(x-3)}}\Bigg| = \frac{A\cancel{(x-3)}}{(x-2)}\Bigg|_{x=3} + \frac{B\,\cancel{(x-3)}}{\cancel{(x-3)}}\Bigg| \quad x=3$$

$$\frac{x+4}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

$$\left(\frac{(x+4)\cancel{(x-2)}}{\cancel{(x-2)}(x-3)}\right)\Bigg|_{x=2} = \left(\frac{A\,\cancel{(x-2)}}{\cancel{(x-2)}}\right) + \frac{B\cancel{(x-2)}}{(x-3)}\Bigg|_{x=2} = 0$$

$$\left.\frac{x+4}{x-3}\right|_{x=2} = A$$

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Exempel 5.
En av termer i nämnarens faktorisering är på formen $x^2 + a^2$ (saknar reella rötter)

EXAMPLE 5 Evaluate $\displaystyle\int \frac{2 + 3x + x^2}{x(x^2+1)}\,dx.$ $\quad 2Q = 3$

Solution Note that the numerator has degree 2 and the denominator degree 3, so no division is necessary. If we decompose the integrand as a sum of two simpler fractions, we want one with denominator x and one with denominator $x^2 + 1$. The appropriate form of the decomposition turns out to be

Method 2

$$\frac{2 + 3x + x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2+1} = \frac{A(x^2+1) + Bx^2 + Cx}{x(x^2+1)}. =$$

$$\begin{aligned} &Ax^2 + A\cdot 1 \\ &Bx^2 \qquad\quad + Cx \\ \hline &(A+B)x^2 + Cx + A \end{aligned}$$

Note that corresponding to the quadratic (degree 2) denominator we use a linear (degree 1) numerator. Equating coefficients in the two numerators, we obtain

$$\begin{cases} A + B = 1 & \text{(coefficient of } x^2) \\ C = 3 & \text{(coefficient of } x) \\ A = 2 & \text{(constant term).} \end{cases}$$

Hence $A = 2$, $B = -1$, and $C = 3$. We have, therefore,

$$\int \frac{2 + 3x + x^2}{x(x^2+1)}\,dx = 2\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{\overset{du}{2x}}{\underset{u}{x^2+1}}\,dx + 3\int \frac{1}{x^2+1}\,dx$$
$$= 2\ln|x| - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x + C.$$

* by the Integration table

Denominators with Repeated Factors

We require one final refinement of the method of partial fractions. If any of the linear or quadratic factors of $Q(x)$ is *repeated* (say, m times), then the partial fraction decomposition of $P(x)/Q(x)$ requires m distinct fractions corresponding to that factor. The denominators of these fractions have exponents increasing from 1 to m, and the numerators are all constants where the repeated factor is linear or linear where the repeated factor is quadratic. (See Theorem 1 below.)

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EXAMPLE 7 Evaluate $\int \frac{1}{x(x-1)^2}\,dx$.

$m = 2$ $\quad$ $2Q = 3$

Method

Solution The appropriate partial fraction decomposition here is

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x^2-2x+1) + B(x^2-x) + Cx}{x(x-1)^2}.$$

Obs:

$$\frac{1}{(x-a)^4} = \frac{\alpha}{(x-a)} + \frac{\beta}{(x-a)^2} + \frac{\gamma}{(x-a)^3} + \frac{\delta}{(x-a)^4}$$

Equating coefficients of x^2, x, and 1 in the numerators of both sides, we get

$$\begin{aligned} A + B &= 0 && \text{(coefficient of } x^2\text{)}\\ -2A - B + C &= 0 && \text{(coefficient of } x\text{)}\\ A &= 1 && \text{(constant term).} \end{aligned}$$

original $\frac{p}{q}$ $= \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2} =$

Hence $A = 1$, $B = -1$, $C = 1$, and

$$\int \frac{1}{x(x-1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$
$$= \ln|x| - \ln|x-1| - \frac{1}{x-1} + C$$
$$= \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C.$$

$\frac{p}{q}$

$A(x^2 - 2x + 1)$
$B(x^2 - x \quad)$
$C(\quad x \quad)$

$(A+B)x^2 + (-2A - B + C)x + A =$

$0x^2 + 0x + 1$

$\boxed{A = 1}$

$A + B = 0 \Rightarrow B = -A = -1$ $\quad$ $\boxed{B = -1}$

$-2A - B + C = 0$ $\quad$ $C = 2A + B = 2(1) - 1 = 1$

$\boxed{C = 1}$

page 347 $\quad$ Case: Quadratic repeating factor

EXAMPLE 8 Evaluate $I = \int \frac{x^2+2}{4x^5+4x^3+x}\,dx$.

$4x^5 + 4x^3 + x =$
$x(4x^4 + 4x^2 + 1)$ $\quad$ $y = x^2$

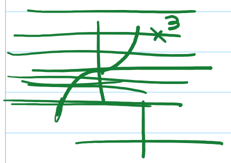
Solution The denominator factors to $x(2x^2+1)^2$, so the appropriate partial fraction decomposition is

$x(4y^2 + 4y + 1)$

$$\frac{x^2+2}{x(2x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}$$
$$= \frac{A(4x^4+4x^2+1) + B(2x^4+x^2) + C(2x^3+x) + Dx^2 + Ex}{x(2x^2+1)^2}.$$

$x(x-a)^2$

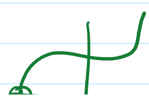
$x^3 + \alpha x^2 + \beta x + \gamma = 0$



Thus

$$\begin{aligned} 4A + 2B &= 0 && \text{(coefficient of } x^4\text{)}\\ 2C &= 0 && \text{(coefficient of } x^3\text{)}\\ 4A + B + D &= 1 && \text{(coefficient of } x^2\text{)}\\ C + E &= 0 && \text{(coefficient of } x\text{)}\\ A &= 2 && \text{(constant term).} \end{aligned}$$

Solving these equations, we get $A = 2$, $B = -4$, $C = 0$, $D = -3$, and $E = 0$.



$I = 2\int \frac{dx}{\ldots} - 4\int \frac{x\,dx}{\ldots} - 3\int \frac{4\ x\,dx}{\ldots}$ $\quad$ Let $u = 2x^2 + 1$.

$A \qquad = 2 \qquad$ (constant term).

Solving these equations, we get $A = 2$, $B = -4$, $C = 0$, $D = -3$, and $E = 0$.

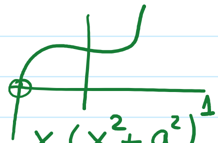
$$I = 2\int \frac{dx}{x} - 4\int \frac{x\,dx}{2x^2+1} - 3\tfrac{1}{4}\int \frac{4\,x\,dx}{(2x^2+1)^2} \qquad \text{Let } u = 2x^2+1,\; du = 4x\,dx$$

$$= 2\ln|x| - \int \frac{du}{u} - \frac{3}{4}\int \frac{du}{u^2}$$

$$= 2\ln|x| - \ln|u| + \frac{3}{4u} + C$$

$$= \ln\left(\frac{x^2}{2x^2+1}\right) + \frac{3}{4}\,\frac{1}{2x^2+1} + C.$$

$\boxed{(2x^2+1)}^{2}$ m



$x(x^2+a^2)^1$

6

The Inverse Trigonometric Substitutions

Three very useful inverse substitutions are:

$$x = a\sin\theta, \qquad x = a\tan\theta, \qquad \text{and} \qquad x = a\sec\theta.$$

These correspond to the direct substitutions:

$$\theta = \sin^{-1}\frac{x}{a}, \qquad \theta = \tan^{-1}\frac{x}{a}, \qquad \text{and} \qquad \theta = \sec^{-1}\frac{x}{a} = \cos^{-1}\frac{a}{x}.$$

The inverse sine substitution

Integrals involving $\sqrt{a^2 - x^2}$ (where $a > 0$) can frequently be reduced to a simpler form by means of the substitution

$$\boxed{x = a\sin\theta} \quad \text{or, equivalently,} \quad \theta = \sin^{-1}\frac{x}{a}.$$

Observe that $\sqrt{a^2 - x^2}$ makes sense only if $-a \le x \le a$, which corresponds to $-\pi/2 \le \theta \le \pi/2$. Since $\cos\theta \ge 0$ for such θ, we have

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$

(If $\cos\theta$ were not nonnegative, we would have obtained $a|\cos\theta|$ instead.) If needed, the other trigonometric functions of θ can be recovered in terms of x by examining a right-angled triangle labelled to correspond to the substitution. (See Figure 6.1.)

$x = a\sin\theta$

$\sin\theta = \dfrac{x}{a}$ $\qquad$ $\tan x = \dfrac{x}{\sqrt{a^2 - x^2}}$

$\cos\theta = \dfrac{\sqrt{a^2 - x^2}}{a}$

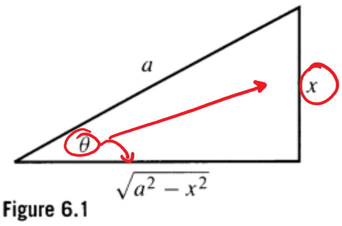


Figure 6.1

Solution Refer to Figure 6.2.

$$\int \frac{1}{(5-x^2)^{3/2}}\,dx \qquad \text{Let } x = \sqrt{5}\sin\theta,\quad dx = \sqrt{5}\cos\theta\,d\theta$$

$$= \int \frac{\cancel{\sqrt{5}}\cancel{\cos\theta}\,d\theta}{5^{3/2}\cos^{\cancel{3}}\theta}$$

$$= \frac{1}{5}\int \sec^2\theta\,d\theta = \frac{1}{5}\tan\theta + C = \frac{1}{5}\,\frac{x}{\sqrt{5-x^2}} + C$$

$$\int \frac{1}{\left(\sqrt{5-x^2}\right)^3}\,dx \qquad 5 = a^2$$

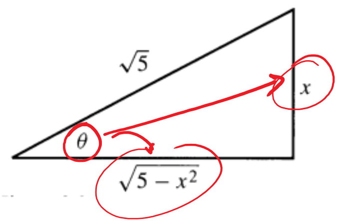
$$5^{3/2} = \left(\sqrt{5}\right)^3$$

$$\frac{\cancel{\sqrt{5}}}{\left(\sqrt{5}\right)^{\cancel{3}}} \qquad \frac{1}{\left(\sqrt{5}\right)^2}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\sec^2\theta = \frac{1}{\cos^2\theta}$$

EXAMPLE 1 Evaluate $\displaystyle\int \frac{1}{(5-x^2)^{3/2}}\,dx$.



$$\sin\theta = \frac{x}{\sqrt{5}} \qquad x = \sqrt{5}\sin\theta$$

$$\cos\theta = \frac{\sqrt{5-x^2}}{\sqrt{5}}$$

$$\tan\theta = \frac{x}{\sqrt{5-x^2}}$$

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The inverse tangent substitution

Integrals involving $\sqrt{a^2+x^2}$ or $\dfrac{1}{x^2+a^2}$ (where $a > 0$) are often simplified by the substitution

$$\boxed{x = a\tan\theta} \quad \text{or, equivalently,} \quad \boxed{\theta = \tan^{-1}\frac{x}{a}.}$$

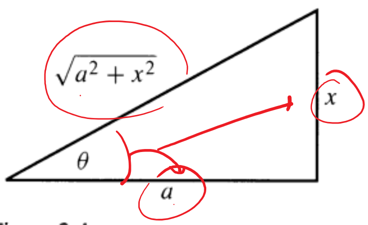
Since x can take any real value, we have $-\pi/2 < \theta < \pi/2$, so $\sec\theta > 0$ and

$$\sqrt{a^2+x^2} = a\sqrt{1+\tan^2\theta} = a\sec\theta.$$

Other trigonometric functions of θ can be expressed in terms of x by referring to a right-angled triangle with legs a and x and hypotenuse $\sqrt{a^2+x^2}$ (see Figure 6.4):

$x^2 + a^2 =$ Hip

Fig 64



$$\boxed{\sin\theta = \frac{x}{\sqrt{a^2+x^2}}}$$

$$\cos\theta = \frac{a}{\sqrt{a^2+x^2}}$$

$$\tan\theta = \frac{x}{a}$$

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EXAMPLE 3 Evaluate (a) $\int \frac{1}{\sqrt{4+x^2}}\,dx$

(a) $\int \frac{1}{\sqrt{4+x^2}}\,dx$ Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$

$= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta$

$= \int \sec\theta\,d\theta \overset{*}{=}$

$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$

$= \ln\left(\sqrt{4+x^2} + x\right) + C_1,$ where $C_1 = C - \ln 2$.

(Note that $\sqrt{4+x^2} + x > 0$ for all x, so we do not need an absolute value on it.)

The tan(θ/2) Substitution

There is a certain special substitution that can transform an integral whose integrand is a rational function of $\sin\theta$ and $\cos\theta$ (i.e., a quotient of polynomials in $\sin\theta$ and $\cos\theta$) into a rational function of x. The substitution is

$$x = \tan\frac{\theta}{2} \quad \text{or, equivalently,} \quad \theta = 2\tan^{-1} x.$$

Observe that

$$\cos^2\frac{\theta}{2} = \frac{1}{\sec^2\frac{\theta}{2}} = \frac{1}{1+\tan^2\frac{\theta}{2}} = \frac{1}{1+x^2},$$

so

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}$$
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{2x}{1+x^2}.$$

Also, $dx = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$, so

$$d\theta = 2\cos^2\frac{\theta}{2}\,dx = \frac{2\,dx}{1+x^2}.$$

The $\tan(\theta/2)$ substitution

If $x = \tan(\theta/2)$, then

$$\cos\theta = \frac{1-x^2}{1+x^2}, \qquad \sin\theta = \frac{2x}{1+x^2}, \quad \text{and} \qquad d\theta = \frac{2\,dx}{1+x^2}.$$

Note that $\cos\theta$, $\sin\theta$, and $d\theta$ all involve only rational functions of x. We examined general techniques for integrating rational functions of x in Section 6.2.

$$\int \frac{1}{2+\cos\theta}\,d\theta \qquad \begin{aligned} &\text{Let } x = \tan(\theta/2), \text{ so} \\ &\cos\theta = \frac{1-x^2}{1+x^2}, \\ &d\theta = \frac{2\,dx}{1+x^2} \end{aligned}$$

$$= \int \frac{\dfrac{2\,dx}{1+x^2}}{2+\dfrac{1-x^2}{1+x^2}} = 2\int \frac{1}{3+x^2}\,dx$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right) + C.$$

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