Föreläsning3.2_alice

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→ ODE with higher order. (Extra)

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for 1st order ODE

I. Non. Homog Linear ODE page 1025

Recall $ay'' + by' + cy = h(x)$ a, b, c constants

solution $y = y_H + y_P$

y_H : solution of $ay'' + by' + cy = 0$

$$\begin{cases} \text{If } r_1 \neq r_2 \in \mathbb{R} & y_H = c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ \text{If } r_1 = r_2 \in \mathbb{R} & y_H = (c_1 + x c_2) e^{r_1 x} \\ \text{If } r_1 = \overline{r_2} \in \mathbb{C} & r_1 = a + bi,\ r_2 = a - bi \\ & y_H = e^{ax}(A\cos bx + B\sin bx) \end{cases}$$

y_P : depends on $h(x)$ & y_H.

Examples

A) $ay'' + by' + cy = h(x)$ $h(x) = k$, $k \in \mathbb{R}$

•) $y_p = Ax + B$
$y_p' = A$
$y_p'' = 0$

$0 + bA + c(Ax + B) = 0x + k$

$$\begin{cases} cA = 0 & \Rightarrow A = 0 \\ bA + cB = k & \Rightarrow B = \frac{k}{c}, \quad c \neq 0. \end{cases}$$

$$\boxed{y_p = \frac{k}{c}}$$

•) $c = 0$ ∴ $ay'' + by' = k$

$y_p = Ax + B$
$y' = A$
$y'' = 0$

$0 + b.A = k$

$A = \frac{k}{b}$ $b \neq 0$.

$$\boxed{y_p = \frac{k}{b} x}$$

•) $c=0$ & $b=0$ $\therefore$ $ay''=k \Rightarrow y''=\frac{k}{a}$, $a\neq 0 \Rightarrow$

$y_p = Ax^2+Bx+C$ $\quad a2A = k \quad \Rightarrow \quad y_p = \frac{k}{2a}x^2 = \frac{k}{a}\left(\frac{x^2}{2}\right)$

$y_p' = 2Ax+B \quad A = \frac{k}{2a}$

$y_p'' = 2A$

$$\boxed{y_p = \frac{k}{a}\frac{x^2}{2}}$$

B) $ay''+by'+cy = h(x) = \text{polynom}$

- $c\neq 0 \Rightarrow y_p = q(x)$, $\partial q(x) = \partial h(x)$
- $c=0$ & $b\neq 0 \Rightarrow y_p = x.q(x)$, $\partial q(x) = \partial h(x)$
- $a\neq 0,\ b=0,\ c=0 \Rightarrow ay'' = h(x) \Rightarrow y'' = \frac{h(x)}{a}$ $\therefore$

$$y' = \int \frac{h(x)}{a}\,dx$$

$$y_p = \int y'\,dx = \iint \frac{h(x)}{a}\,dx^2.$$

Exemple of B.

$*$ $y''-y' = x^2$

y_H: $r^2-r=0 \Rightarrow r(r-1)=0 \Rightarrow r_1=0$, $r_2=1$

$y_H = C_1e^{0x}+C_2e^{1x} = C_1+C_2e^{x}$

$*$ $C=0,\ b\neq 0$ $\therefore$ $q(x) = Ax^2+Bx+C$

$y_p = x\,q(x) = Ax^3+Bx^2+Cx$

$y_p = Ax^3+Bx^2+Cx$

$y_p' = 3Ax^2+2Bx+C$

$y_p'' = 6Ax+2B$

$$\begin{array}{l} 1(\qquad 6Ax+2B) \\ -1(\ 3Ax^2+2Bx\ +C\ ) = 1x^2+0x+0 \\ \hline -3Ax^2+(6A-2B)x+2B-C = 1x^2+0x+0 \end{array}$$

$$\begin{cases} -3A = 1 & \boxed{A = -\frac{1}{3}} \\ 6A-2B=0 & \cancel{2}B = \cancel{6}A \quad B=3A \quad \boxed{B=-1} \\ 2B-C=0 & 2B = C \quad \boxed{C=-2} \end{cases}$$

$y_p = -\frac{1}{3}x^3-x^2-2x$

$y = y_H+y_p = C_1+C_2e^{x}-\frac{1}{3}x^3-x^2-2x$

C) $ay''+by'+cy = h(x) = g(x)e^{\alpha x}$ with $g(x)$ = polynomial. (1)

C) $ay'' + by' + cy = h(x) = g(x)\, e$ with $g(x)$ = polynomial. (1)

Goal: Transform C) in B) considering: $\boxed{y_p = z(x)\, e^{\alpha x}}$

- substitute y_p into eq (1)
- find a new equation for $z(x)$

Example: $y'' + 3y' + 2y = x\, e^{-x} = g(x)\, e^{\alpha x}$

y_H: $r^2 + 3r + 2 = 0$

$r_{1,2} = \dfrac{-3 \pm \sqrt{9 - 4.1.2}}{2}$

$r_{1,2} = \dfrac{-3 \pm 1}{2}$

$r_{1,2} = -1, -2$

$y_H = \underline{c_1 e^{-x}} + c_2 e^{-2x}$

Assume $y_p = z(x)\, e^{-x}$

$y_p' = z'(x)\, e^{-x} + z(x)(-1)\, e^{-x}$

$y_p'' = z''(x)\, e^{-x} - \underbrace{z'(x) e^{-x}} - z'(x)\, e^{-x} + z(x)\, e^{-x}$

$$
\begin{aligned}
y_p'' &= (z'' - 2z' + z)\, e^{-x} \\
+3y_p' &= 3(\quad z' - z)\, e^{-x} \\
+2y_p &= 2(\quad z)\, e^{-x} \\
\hline
x\, \cancel{e^{-x}} &= (z'' + z' + 0z)\, \cancel{e^{-x}}
\end{aligned}
$$

New equation: $z'' + z' = x$ (2) (this is case C) with $c = 0$, $b \neq 0$

so $z_p = x(Ax + B) = Ax^2 + Bx$

$z_p' = 2Ax + B$

$z_p'' = 2A$

substituting in eq (2)

$$2A + 2Ax + B = 1x + 0$$

$$2Ax + (2A + B) = 1x + 0 \Rightarrow$$

$$\begin{cases} 2A = 1 & \Rightarrow \quad A = 1/2 \\ 2A + B = 0 & \Rightarrow \quad B = -2A = -2\left(\frac{1}{2}\right) = -1 \end{cases}$$

$$z_p = \frac{1}{2}x^2 - 1x$$

$y_p = \left(\frac{1}{2}x^2 - 1x\right) e^{-x}$

$\quad{}^{x} \quad {}^{2x} \quad\quad {}^{-x}$

$$y = y_H + y_P = C_1 e^{-x} + C_2 e^{-x} + \left(\tfrac{1}{2}x^2 - 1x\right)e^{-x}$$

II. §18.5 page 1023 Euler (Equidimensional) Equation

(3) $a x^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$

2^{nd} order
linear Homogeneous ODE
<u>Non constant coefficients</u>

<u>Hypothesis</u> for $x>0$, we assume the solution on the form:

$$y = x^r$$
$$y' = r x^{r-1}$$
$$y'' = r(r-1)x^{r-2}$$

Substitute into (3)

$$a r(r-1)x^2 . x^{r-2} + b r x x^{r-1} + c x^r = 0$$

$$a(r^2-r)x^r + brx^r + cx^r = 0$$

$$\left(a(r^2-r) + br + c\right)\underbrace{x^r}_{\neq 0} = 0 \Rightarrow \boxed{ar^2 + (b-a)r + c = 0} \; (*)$$

r_1, r_2 are roots of $(*)$

- $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$

$$y = C_1 x^{r_1} + C_2 x^{r_2} \qquad (x>0)$$

(general solution: $(x \neq 0)$
$y = C_1 |x|^{r_1} + C_2 |x|^{r_2}$.

- $r_1 = r_2 \in \mathbb{R}$

$$y = C_1 x^r + C_2 x^r . \ln x \qquad (x>0)$$

(or general case $(x \neq 0)$
$y = C_1 |x|^{r_1} + C_2 |x|^r . \ln|x|$

- $r_1 = \overline{r_2}$; $r_1, r_2 \in \mathbb{C}$ $\quad r_1 = a+bi$; $r_2 = a-bi$

$$x^{a\pm bi} = e^{\ln(x^{a\pm bi})} = e^{(a\pm bi)\ln x} = e^{(a\ln x \pm b\ln x\, i)} \underset{\text{Moivres formula}}{=}$$

$$= e^{a\ln x}\left(\cos(b\ln x) + i\sin(b\ln x)\right)$$

$$= x^a \left(\cos(b \ln x) \pm i \sin(b \ln(x)) \right)$$

And the solution is given as:

$$y = c_1 |x|^a \cos(b \ln|x|) + c_2 |x|^a \sin(b \ln|x|)$$

III. § 18.6 page 1029

2^{nd} order ODE with Nonconstant coeff (general case)

$$a_2(x)\, y'' + a_1(x) y' + a_0(x) y = f(x) \qquad (4)$$

Hypothesis

1) Assume we know that $y_1(x)$ & $y_2(x)$ are solutions of the Homogeneous equation

2) Assume $y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$

Derivate: $y_p' = u_1' \cdot y_1 + u_1 y_1' + u_2' \cdot y_2 + u_2 y_2'$

3) Assume $u_1' y_1 + u_2' y_2 = 0$

So $y_p' = u_1 y_1' + u_2 y_2' \qquad (*)$

$$y_p'' = u_1' y_1' + u_1 \cdot y_1'' + u_2' y_2' + u_2 y_2''$$

4) Substitute y_p, y_p' $(*)$, y_p'' into eq (4)

$$a_2(x) \left(u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' \right)$$
$$+\ a_1(x) \left(\qquad + u_1 y_1' \qquad + u_2 y_2' \right)$$
$$+\ a_0(x) \left(\qquad u_1 y_1 \qquad + u_2 y_2 \right)$$

$$u_1 \underbrace{\left(a_2 y_1'' + a_1 y_1' + a_0 y_1 \right)}_{=0} + u_2 \underbrace{\left(a_2 y_2'' + a_1 y_2' + a_0 y_2 \right)}_{=0}$$

y_1 and y_2 are solutions of the Homogeneous equation.

$+ a_2 (u_1' y_1' + u_2' y_2') \qquad = f(x)$

$$+ a_2(u_1' y_1' + u_2' y_2') = f(x).$$

are solutions of the Homogeneous equation.

So the 2 equations left are:

$$\circledast \quad \begin{cases} u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x) = 0 \\ u_1'(x)\, y_1'(x) + u_2'(x)\, y_2'(x) = \dfrac{f(x)}{a_2(x)} \end{cases}, \quad a_2(x) \neq 0.$$

$$\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{f(x)}{a_2(x)} \end{bmatrix}$$

By Cramer's Rule (page 615, theo 6 , § 10.7)

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{f(x)}{a_2(x)} & y_2' \end{vmatrix}}{W} \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f(x)}{a_2(x)} \end{vmatrix}}{W}$$

W = Wronskian $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

So $u_1' = \dfrac{-y_2 \frac{f(x)}{a_2(x)}}{W}$ $\qquad u_2' = \dfrac{y_1 \frac{f(x)}{a_2(x)}}{W}$

$$u_1 = \int u_1'\, dx \qquad u_2 = \int u_2'\, dx$$

<u>Example</u>: $y'' + y = \tan x$

1) Solve the homog: $y'' + y = 0$ $\quad r^2 + 1 = 0$ $\quad r_{1,2} = \pm i$

$$y_H = C_1 \cos x + C_2 \sin x$$

2) y_p is obtained by the Method of Variation of Parameters:

Hypothesis: $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

1st restriction: $u_1' \cdot y_1 + u_2' y_2 = 0$

$u_1' \cos x + u_2' \sin x = 0$

1st restriction: $u_1 \cdot y_1 + u_2 y_2 = 0$

$$\boxed{u_1' \cos x + u_2' \sin x = 0}$$

& according to the final system $**$, eq (2): $u_1' y_1' + u_2' y_2' = \frac{f(x)}{a_2(x)}$

$$u_1'(-\sin x) + u_2' \cos x = \frac{\tan x}{1}$$

Now solve the system

$$\begin{cases} u_1' \cos x + u_2' \sin x = 0 \\ u_1'(-\sin x) + u_2' \cos x = \tan x \end{cases}$$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \tan x \end{bmatrix}$$

$$u_1' = \frac{-\sin x \tan x}{W} \qquad u_2' = \frac{\cos x \tan x}{W}$$

$W = \cos^2 x + \sin^2 x = 1$

So $u_1' = -\frac{\sin^2 x}{\cos x}$ $\qquad u_2' = \sin x$

To find u_1 & u_2 we have to integrate: u_1' & u_2'.

$u_1 = -\int \frac{\sin^2 x}{\cos x} dx \overset{*}{=}$ $\qquad$ * Trigonometric Relations: $\sin^2 x = 1 - \cos^2 x$

$-\sin^2 x = \cos^2 x - 1$

$u_1 = \int \frac{\cos^2 x - 1}{\cos x} dx = \int \left(\cos x - \frac{1}{\cos x}\right) dx \overset{*}{=}$ $\qquad$ * Tabel

$$\boxed{u_1 = \sin x - \ln(\sec x + \tan x)}$$

$u_2(x) = \int \sin x \, dx = -\cos x$

So $y_p = \left(\sin x - \ln(\sec x + \tan x)\right) \cdot \cos x +$

$(-\cos x) \cdot \sin x =$

$$y_p = -\ln(\sec x + \tan x)$$

$$\boxed{y = y_H + y_p = c_1 \cos x + c_2 \sin x - \ln(\sec x + \tan x)}$$

IV ODE of higher order.

$$y^{(n)} + \alpha_{n-1} y^{(n-1)} + \dots + \alpha_1 y' + \alpha_0 y = h(x) \qquad \text{order } n.$$

Same procedure: $y = y_H + y_p$

Hypothesis 1: $y_H = e^{rx} \Rightarrow p(r) = r^n + \alpha_{n-1} r^{n-1} + \dots + \alpha_1 r + \alpha_0 = 0$

↳ Characteristic polynomial.

↳ find the roots.

If w is a root of $p(r)$ with multiplicity equal to k

So it's contribution to the homogeneous solution y_H is:

$$y_w = (C_0 + C_1 x + \dots + C_{k-1} x^{k-1}) e^{wx}$$

And again $y_H = y_w + D_j e^{r_j x} + \dots D_n e^{r_n x}$ (if $r_j, \dots r_n$ are simple roots)

Example:

$$y^{(4)} - 2y^{(3)} + 2y' - y = x e^x$$

i) $y = y_H + y_p$

ii) $y_H = e^{rx} \Rightarrow p(r) = r^4 - 2r^3 + 2r - 1 = 0 \quad \circledast$

$p(r) = (r-1)^3 \cdot (r+1)$ → $r_1 = r_2 = r_3 = 1$, $r_4 = -1$

So $y_H = (C_0 + C_1 x + C_2 x^2) e^x + D e^{-x}$

iii) $y_p = z(x)e^{\sim}$ as we did for 2nd order ODE.

$\vdots$