Mål: Linear Algebra   Bok: Lay.

$V = \{\vec{v} \,/\, \vec{v} = (a_1, \dots, a_n),\ a_i \in \mathbb{R}\}$
vector space.
$\vec{v}_1, \vec{v}_2 \in V$ ∴
$\vec{v}_1 + \vec{v}_2 \in V$
$\lambda \vec{v}_1 \in V$

Let's recall the system from Tentan.

$$\begin{cases} 1x_1 + 2x_2 + 3x_3 = 6 \\ 0x_1 + 1x_2 - 1x_3 = 0 \\ -2x_1 + 3x_2 + 0x_3 = 1 \end{cases} \iff \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ -2 & 3 & 0 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}}_{b}$$

$$A \cdot X = b$$

Matricial form of the system

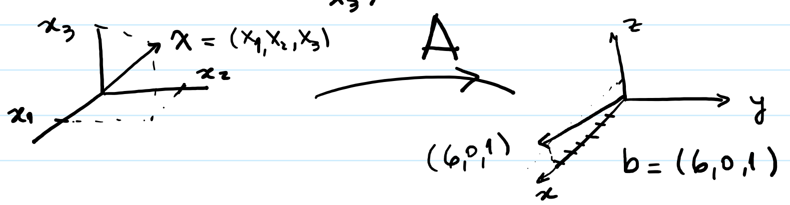
Now let's writte the system in another way:

$$x_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}_{u_1} + x_3 \underbrace{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}}_{u_2} + x_3 \underbrace{\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}}_{u_3} = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}$$

$V = \mathbb{R}^3$ vector space.

- sum of vectors with 3 coordinates is another vector with 3 coordinates.
- $\lambda \vec{v} \in \mathbb{R}^3$ if $\vec{v} \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$.

Obs 1. when we solve the system we want to find $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ = vector $\in \mathbb{R}^3$ such that



$$[A]X = b$$

Making an analogy with functions: $A(x) = b$
$f(x) = b$

we are surching for an element x ($X \in \mathbb{R}^3$) on the domain of f such that $f(x) = b$

Or: we are asking: Does b belong to the $\mathrm{Im}(f)$?
(Does the system has solution?)

Obs 2: We can do another question involving concepts of linear algebra which are related to the same procedure of solving $Ax = b$.

Definition:

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given the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n \in \mathbb{R}^n$ & the constants $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$,

the vector $\vec{y} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n = \sum_{k=1}^{n} \alpha_k \vec{v}_k$

is called a <u>linear combination</u> of $\vec{v}_1, \dots, \vec{v}_n$.

So

Question: when is a vector $\vec{y} \in \mathbb{R}^n$ a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$?

Answer: when we can find $\alpha_1, \alpha_2 \dots \alpha_n$ constants such that $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n = \vec{y}$

Let's do this question for our case in $\mathbb{R}^3$.

$$\vec{y} = b = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}$$

Is $\vec{y}$ a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ with

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \ ?$$

Are there constants $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix} \ ?$$

Can we solve the system

$$\begin{cases} 1\alpha_1 + 2\alpha_2 + 3\alpha_3 = 6 \\ 0\alpha_1 + 1\alpha_2 - 1\alpha_3 = 0 \\ -2\alpha_1 + 3\alpha_2 + 0\alpha_3 = 1 \end{cases} \quad ?$$

YES! And in this case the solution is unique!

So the vector $\vec{y} = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}$ can be uniquely

written as $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$

Obs: we are going to study this unicity in more details

Question: ⊛ Can $\vec{0}$ be written as a linear combination of $\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$?

Yes! allways because $0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_n = \vec{0}$

⊛ But, are there other constants? other possibilities to generate the vector $\vec{0}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$?

Answer:

- yes: Then we say $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ are LD ≡ linearly Dependent.
- No: The we say $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ are LI ≡ linearly independent.

Let's see one example & take further conclusions.

$$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}_{\vec{u}_1}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \right\} \quad \text{LI or LD?}$$

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{LI or LD?}$$

Solution:

(I)
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ -2 & 3 & 0 & 0 \end{array}\right] L_3 \leftarrow L_3 + 2L_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 7 & 6 & 0 \end{array}\right) L_3 \leftarrow L_3 - 7L_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right) \begin{array}{l} L_1 \leftarrow L_1 - 3\tilde{L}_3 \\ L_2 \leftarrow L_2 + \tilde{L}_3 \\ \tilde{L}_3 \leftarrow \frac{1}{13} L_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) L_1 \leftarrow L_1 - 2L_2$$

Echolon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

(II)
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ -2 & 3 & 1 & 0 \end{array}\right] L_3 \leftarrow L_3 + 2L_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 7 & 7 & 0 \end{array}\right) L_3 \leftarrow L_3 - 7L_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) L_1 \leftarrow L_1 - 2L_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Echolon form.

Let's solve the system:

Echolon form
$$\left(\begin{array}{ccc|c} 1 & & & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{cases} 1\alpha_1 + 0 + 0 = 0 \\ 0 + 1\alpha_2 + 0 = 0 \\ 0 + 0 + 1\alpha_3 = 0 \end{cases}$$

So $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is the unique solution of this system.

So $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is LI.

Let's solve the system:

$$\begin{cases} 1\alpha_1 + 0\alpha_2 + 1\alpha_3 = 0 \\ 0 + \alpha_2 + \alpha_3 = 0 \\ 0 = 0 \end{cases}$$

$\alpha_1 = -\alpha_3$
$\alpha_2 = -\alpha_3$
α_3 is free.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -\alpha_3 \\ -\alpha_3 \\ \alpha_3 \end{pmatrix} = \alpha_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \quad \alpha_3 \in \mathbb{R}$$

So, there is INFINITE solutions or INFINITE WAYS TO GENERATE ZERO AS LINEAR COMBINATION OF $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$.

So the set of vectors is LD.

? Is the system (II) telling us something else?

YES!

$$\left(\begin{array}{cc|c|} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

This tells us that

$$1\bar{v}_1 + 1\bar{v}_2 = \bar{v}_3$$

So $\bar{v}_3$ was itself a linear combination of $\{\bar{v}_1, \bar{v}_2\}$

And therefore $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is LD.