Vector Space $= \boxed{V, +, \lambda\cdot}$

$V=$ set of vectors

$+ =$ sum operation $\quad \bar{u}, \bar{v} \in V,\ \bar{u}+\bar{v} \in V$

$\lambda\cdot =$ multiplication by scalar $\quad \lambda \in \mathbb{R},\ \bar{u} \in V,\ \lambda\bar{u} \in V$

Properties:

$$\begin{cases} \vec{u}+\vec{v} = \bar{v}+\bar{u} \\ (\bar{u}+\bar{v})+\bar{w} = u+(\bar{v}+\bar{w}) \\ \bar{u}+\vec{0} = \vec{u} \\ \vec{u}+(-\vec{u}) = \vec{0} \end{cases}$$

$$(\lambda+\beta)\vec{u} = \lambda\bar{u}+\beta\vec{u}$$
$$\lambda(\beta\bar{u}) = (\lambda\beta)\vec{u}$$
$$1(\bar{u}) = \bar{u}$$
$$\lambda(\bar{u}+\bar{v}) = \lambda\bar{u}+\lambda\bar{v}$$

MVE465

$\underbrace{V = \mathbb{R}^2, \mathbb{R}^3}$

Analitic Geo.

$$\boxed{V = \mathbb{R}^n}$$

$V = M_{n\times m} = \{A_{n\times m},\ a_{ij} \in \mathbb{R}\}$

$\Updownarrow$

$\mathbb{R}^{n\times m} = \{\bar{v} = (a_{11}, a_{12}, \dots, \dots a_{nm})$

$V = \mathcal{P}_n = \{p(x),\ \partial p(x) \leq n\}$

$p(x) = \underbrace{a_n x^n + \dots + a_1 x + a_0}$

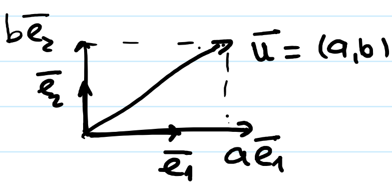
$\Updownarrow$

$\mathbb{R}^{n+1} = \{\bar{v} = \underbrace{(a_n, a_{n-1}, \dots, a_1, a_0)}_{n+1}\}$

Example

$V = \mathbb{R}^2$

$$\bar{u} = (a,b) = a(1,0) + b(0,1)$$
$$= a\bar{e}_1 + b\bar{e}_2$$
$$= a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$$
$$= \boxed{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}}$$

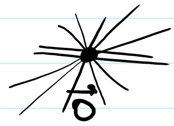


Goal: LD, LI, Basis

LD = linearly Dependent

$V = \mathbb{R}^2$ (Example)

$\{\vec{0}\}$ $\quad \underbrace{\lambda \vec{0} = \vec{0}}$, $\lambda \in \mathbb{R}$ (infinit solutions)



→ The set $\{\vec{0}\}$ is Linearly Dependent $\equiv$ LD

→ $\{\vec{u}\}$ $\quad \vec{u} \neq \vec{0}$

$\vec{0}$

The set $\{\vec{u}\}$ is LI.
since $\vec{u} \neq \vec{0}$

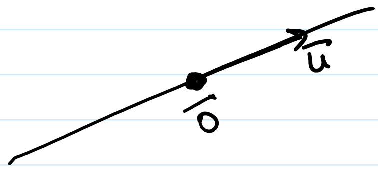
$\lambda \vec{u} = \vec{0} \Rightarrow \boxed{\lambda = 0}$ unique solution

$\vec{u} \neq \vec{0}$

→ $\{\vec{u}, \vec{0}\}$ $\quad$ ⊛ $\boxed{\alpha \vec{u} + \beta \vec{0} = \vec{0}}$ $\qquad \vec{u} = (1, 2)$

This is again a LD set

$$\alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since we have infinite solutions ⊛

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right] L_2 \leftarrow L_2 - 2L_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$1\alpha + 0\beta = 0 \quad \Rightarrow \boxed{\alpha = 0}$

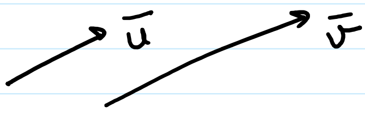
$0\alpha + 0\beta = 0 \quad \Rightarrow \beta =$ is free

$\boxed{\text{infinite solutions}}$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \underbrace{\beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{solution}}$$

$\{\vec{u}, \vec{v}\}$ $\quad \vec{u} \neq \vec{0}$, $\vec{v} \neq \vec{0}$

if $\vec{v} \parallel \vec{u} \Leftrightarrow \vec{v} = \lambda \vec{u}$ $\boxed{\{\vec{u}, \vec{v}\} \text{ LD}}$



$\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\{\vec{u}, \vec{v}\} = \text{LD}$

$$\lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \lambda \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}\lambda\\ \beta\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

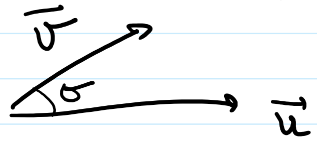
$$\boxed{\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}\lambda\\ \beta\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}}$$

$1\lambda + 2\beta = 0$ $\quad \boxed{\lambda = -2\beta}$ $\quad \beta$ is free

infinite solutions

$\bar{u}$ is NOT parallel to $\bar{v}$

$\{\bar{u}, \bar{v}\}$ LI if $\bar{u} \nparallel \bar{v} \iff \bar{u} \neq \lambda \bar{v}$



$$\lambda \bar{u} + \beta \bar{v} = \vec{0}$$

$$\lambda \begin{pmatrix}1\\ 2\end{pmatrix} + \beta \begin{pmatrix}-1\\ 3\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0\\ 2 & 3 & 0\end{array}\right] \quad L_2 \leftarrow L_2 - 2L_1$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0\\ 0 & \underbrace{3-2(-1)}_{5} & 0\end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0\\ 0 & \cancel{5}\,1 & 0\end{array}\right] \quad \begin{array}{l} L_1 \leftarrow L_1 + L_2\\ L_2 \leftarrow L_2/5\end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0\\ 0 & 1 & 0\end{array}\right] \Rightarrow \begin{array}{l} \lambda \cdot 1 + 0\beta = 0 \Rightarrow \boxed{\lambda = 0}\\ 0 \cdot \lambda + 1\beta = 0 \Rightarrow \boxed{\beta = 0}\end{array}$$

In this case $\{\bar{u}, \bar{v}\} = \{\begin{pmatrix}1\\ 2\end{pmatrix} \begin{pmatrix}-1\\ 3\end{pmatrix}\} = \underline{\text{LI}}$

$\{\bar{u}, \bar{v}, \bar{w}\}$ in $\mathbb{R}^2$

$\{\bar{u}, \bar{v}, \bar{0}\}$ LD

$\{\bar{u}, \bar{v}, \bar{w}\} \longrightarrow \begin{bmatrix} 1 & -1 & a\\ 2 & 3 & b\end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & *\\ 0 & 1 & *\end{bmatrix}$

$\bar{w} = \begin{pmatrix}-2\\ 4\end{pmatrix}$ $\qquad \bar{u} = \begin{pmatrix}1\\ 2\end{pmatrix}$ $\bar{v} = \begin{pmatrix}-1\\ 3\end{pmatrix}$ $\bar{w} = \begin{pmatrix}-2\\ 4\end{pmatrix}$

Question: $\{\bar{u}, \bar{v}, \bar{w}\}$ ~~LI~~ or LD?

$\alpha \bar{u} + \beta \bar{v} + \gamma \bar{w} = \vec{0}$ $\quad$ Has 1 uniq solution? LI or Has infinite sol? LD

$$\alpha \begin{pmatrix}1\\ 2\end{pmatrix} + \beta \begin{pmatrix}-1\\ 3\end{pmatrix} + \gamma \begin{pmatrix}-2\\ 4\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\alpha\begin{pmatrix}1\\2\end{pmatrix}+\beta\begin{pmatrix}-1\\3\end{pmatrix}+\gamma\begin{pmatrix}-2\\4\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & -1 & -2\\2 & 3 & 4\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\left[\begin{array}{ccc|c}\boxed{1} & -1 & -2 & 0\\2 & 3 & 4 & 0\end{array}\right] L_2 \leftarrow L_2-2L_1$$

$$\left[\begin{array}{ccc|c}1 & -1 & -2 & 0\\0 & 3-2(-1) & 4-2(-2) & 0\end{array}\right]\to\left[\begin{array}{ccc|c}1 & -1 & -2 & 0\\0 & 1 & 8/5 & 0\end{array}\right] L_1\leftarrow L_1+L_2$$

$$\to\left[\begin{array}{ccc|c}1 & 0 & -2/5 & 0\\0 & 1 & 8/5 & 0\end{array}\right]$$

$$\begin{bmatrix}1 & 0 & -2/5\\0 & 1 & 8/5\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$1\alpha+0\beta-2/5\gamma=0 \Longrightarrow 1\alpha=2/5\gamma$$

$$0\alpha+1\beta+8/5\gamma=0 \Longrightarrow 1\beta=8/5\gamma$$

$$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}2/5\gamma\\8/5\gamma\\\gamma\end{pmatrix}=\gamma\begin{pmatrix}2/5\\8/5\\1\end{pmatrix}$$

γ is free

Infinite solutions $\Longrightarrow$ LD

1°) $\{\bar{u},\bar{v},\bar{w}\}$ in $\mathbb{R}^2$

ex $\left\{\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}-1\\3\end{pmatrix}\underbrace{\begin{pmatrix}-2\\4\end{pmatrix}}_{\bar{w}}\right\}$ LD

$\{\bar{u},\bar{v}\}$ LI

$\{\bar{u},\bar{v},\bar{w}\}$

$$\left[\begin{array}{cc|c}1 & 0 & -2/5\\0 & 1 & 8/5\end{array}\right]$$

$$-\frac{2}{5}\begin{pmatrix}1\\2\end{pmatrix}+8/5\begin{pmatrix}-1\\3\end{pmatrix}=\begin{pmatrix}-2/5-8/5\\-\frac{4}{5}+\frac{24}{5}\end{pmatrix}=$$

$$=\begin{pmatrix}-10/5\\+20/5\end{pmatrix}=\begin{pmatrix}-2\\4\end{pmatrix}$$

$$= \begin{pmatrix} -10/5 \\ +20/5 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

Given $\{\bar{u}, \bar{v}\}$, can $\bar{w} = (-2, 4)$ be written
$\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right\}$ as a linear combination of $\bar{u}$ & $\bar{v}$?

? $\alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$?

? α, β ?

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \leftrightarrow [A|B] \leftrightarrow \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 2 & 3 & 4 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & -2/5 \\ 0 & 1 & 8/5 \end{array}\right]$$

unique solution

$\bar{w}$ is a linear comb. of $\bar{u}$ & $\bar{v}$

$$\bar{w} = -\tfrac{2}{5}\bar{u} + \tfrac{8}{5}\bar{v}$$

$\{\bar{u}, \bar{v}\}$ LI $\quad \boxed{\alpha \bar{u} + \beta \bar{v} = \vec{0}}$ has unique solution
$\alpha = 0, \beta = 0$ (trivial solution)

$\{\bar{u}, \bar{v}\}$ LI is also a generator of V

because for all choices of $\bar{w} \in V$

$\bar{w}$ can be written as linear combination of $\{\bar{u}, \bar{v}\}$

$\alpha \bar{u} + \beta \bar{v} = \bar{w}$ has also unique solution.

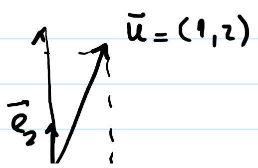
$$\bar{w} = (\alpha, \beta)_{\{\bar{u}, \bar{v}\}}$$

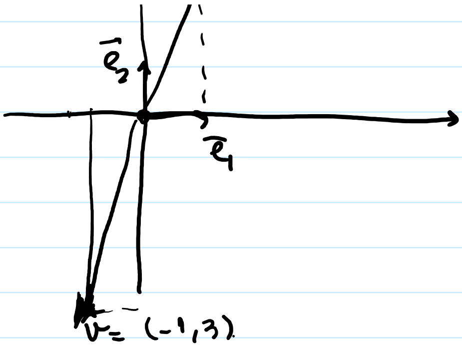
$$\begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} | \\ w \\ | \end{bmatrix}$$

This Non Homogeneous system has unique solution.

$$AX = B \qquad X = \boxed{X}_H + \boxed{X}_P$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$





$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$

$\boxed{\{\bar{e}_1, \bar{e}_1\} \text{ is a basis for } V}$

$\to$ $\{\bar{e}_1, \bar{e}_2\}$ LI

$\to$ $\{\bar{e}_1, \bar{e}_1\}$ is a generator

any vector $\bar{w} \in V$ can be written as a linear comb. of $\bar{e}_1, \bar{e}_2$

$\bar{w} = (-2, 4) = -2\,\bar{e}_1 + 4\,\vec{e}_2$

$\bar{w} = (x, y) = x\,\bar{e}_1 + y\,\vec{e}_2$

$\boxed{\text{Span}\{\bar{e}_1, \bar{e}_2\} = V = \mathbb{R}^2}$

LI

$\{\bar{u}, \bar{v}, \bar{w}\}$ $\quad \boxed{\bar{w} = \alpha\,\bar{u} + \beta\,\bar{v}}$

$\to$ This set HAS TO BE LD

To prove that $\{\bar{u}, \bar{v}, \bar{w}\}$ is LD we start with

$a\,\bar{u} + b\,\bar{v} + c\,\bar{w} = \vec{0}$? $\quad a, b, c$ are unique ?

$a\bar{u} + b\bar{v} + c\,(\alpha\,\bar{u} + \beta\bar{v}) = \vec{0}$

$(\underbrace{a + c\alpha})\vec{u} + (\underbrace{b + c\beta})\,\bar{v} = 0$ since by Hypothesis $\{\bar{u}, \bar{v}\}$ LI

$\begin{cases} a + c\alpha = 0 \Rightarrow a = c.\alpha \\ b + c\beta = 0 \Rightarrow b = c.\beta \end{cases}$ $\quad$ c ? $\quad$ c ?

? c = ? is free !

$(a, b, c) = (\alpha c, \beta . c, c) = c\,(\alpha, \beta, 1)$ $\quad$ c free in $\mathbb{R}$

$c \in \mathbb{R}$ constant.

$V, +, \lambda$

$$\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\text{• Basis of } V}$ $\left\{ \begin{array}{l} B = \{ b_1, b_2, \dots b_n \} \text{ LI} \\ \text{Span}\{b_1, \dots, b_n\} \text{ is } V \end{array} \right.$

$$\underbrace{\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix}}\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{\bar{w}}$$

$$\text{Span}\{b_1, \dots b_n\} = \{ \bar{u} = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n \mid \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}, \dots \alpha_n \in \mathbb{R} \}$$

given $\bar{w} \in V$ $\quad \bar{w} = (\alpha_1, \alpha_2, \dots \alpha_n)_B$ coordinates of $\vec{w}$ according to B.

$$\bar{w} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (-2, 4)_{\mathcal{E}}$$

$\mathcal{E} = \{ \bar{e}_1, \bar{e}_2 \}$ the standard basis

$$\bar{w} = \begin{pmatrix} -2/5 \\ 8/5 \end{pmatrix}_B = -\frac{2}{5}\bar{u} + \frac{8}{5}\bar{v}$$

$\bar{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \bar{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$B = \{ \bar{u}, \bar{v} \}$ for $V = \mathbb{R}^2$

Practical Example (Canvas Material F 4.2

$V = \mathbb{R}^3$ $\quad \bar{u} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$ $\quad \bar{v} = \begin{pmatrix} -6 \\ 1 \\ 7 \end{pmatrix}$ $\quad \bar{w} = \begin{pmatrix} 0 \\ -5 \\ 2 \end{pmatrix}$ $\quad \bar{z} = \begin{pmatrix} 3 \\ 7 \\ -5 \end{pmatrix}$

a) $K_1 = \{ \bar{u} \}$ LI $\qquad \text{Span}\{\bar{u}\} = \{ \bar{t} = \lambda \bar{u} \}$

$\{ t = \lambda(3, 2, -4) \}$

$K_1^v = \{ \bar{v} \}$ LI

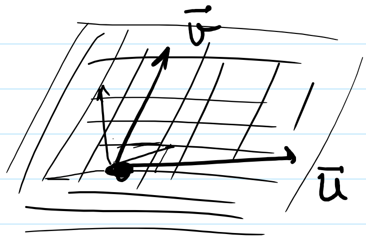
$\text{Span}\{\bar{v}\} = \{ \bar{t} = \beta \bar{v} \}$

the line passing through the origin $O = (0,0,0)$ with $\bar{v}$ as direction vector.

b) $K_2 = \{\bar{u}, \bar{v}\} = \left\{ \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 7 \end{pmatrix} \right\}$ $\quad \bar{u} \neq \lambda \vec{v}$

<u>K_2 is LI set</u>

$\text{Span}\{\bar{u}, \bar{v}\} = \{ \vec{t} = \lambda \bar{u} + \beta \bar{v} \}$
$\lambda, \beta \in \mathbb{R}$



Plane passing through zero
with direction vectors $\bar{u}, \bar{v}$

$b_2)$ $K_2^{(u,w)} = \{\bar{u}, \bar{w}\} = \left\{ \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ 2 \end{pmatrix} \right\}$

$\bar{w}$ = can be written as a linear comb. of $\bar{u}, \bar{w}$?

$$\left[\begin{array}{cc|c|c} 1 & 0 & 0 & 0 \\ 2 & -5 & 0 & -5 \\ -4 & 2 & 0 & 2 \end{array}\right] \begin{array}{l} L_1 \leftarrow \frac{1}{3} L_1 \\ L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 + 4L_1 \end{array}$$

$$\left[\begin{array}{cc|c|c} 1 & 0 & 0 & 0 \\ 0 & \cancel{-5}_{-5}\ 1 & 0 & \cancel{-5}_{-5}\ 1 \\ 0 & \cancel{2}_{2} = 1 & 0 & \cancel{2}_{2}\ 1 \end{array}\right] \begin{array}{l} \\ L_2 \leftarrow -\frac{1}{5} L_2 \\ L_3 \leftarrow \frac{1}{2} L_3 \end{array}$$

$$\underbrace{\left[\begin{array}{cc|c|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{LI}}$$

$\underline{\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$ $\quad \alpha = 0, \beta = 0$ unique solution

$\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \boxed{\begin{array}{l} \alpha = 0 \\ \beta = 1 \end{array}}$

<u>$V = \mathbb{R}^3$</u> $\{\bar{u}, \bar{v}\}$ LI $\quad$? $\underbrace{\{\bar{u}, \bar{v}, \bar{w}, \vec{z}\}}$ $\underline{\text{~~LI~~ or LD}}$?