Mål: $T: \mathbb{R}^n \to \mathbb{R}^m$ linear Transform

* Classification : 1:1
ONTO
Bijection $\Leftrightarrow$ T is invertible
$\boxed{T^{-1} \text{ exists}}$

* How to compute & understand T^{-1}!

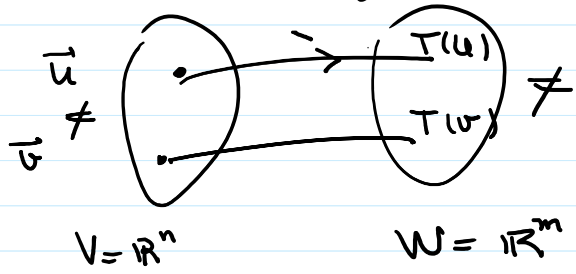
$$T: \mathbb{R}^n \to \mathbb{R}^m$$
$$\vec{u} \to T(\vec{u}) = A_{m\times n}[u]_{n\times 1} = [T(u)]_{m\times 1}$$

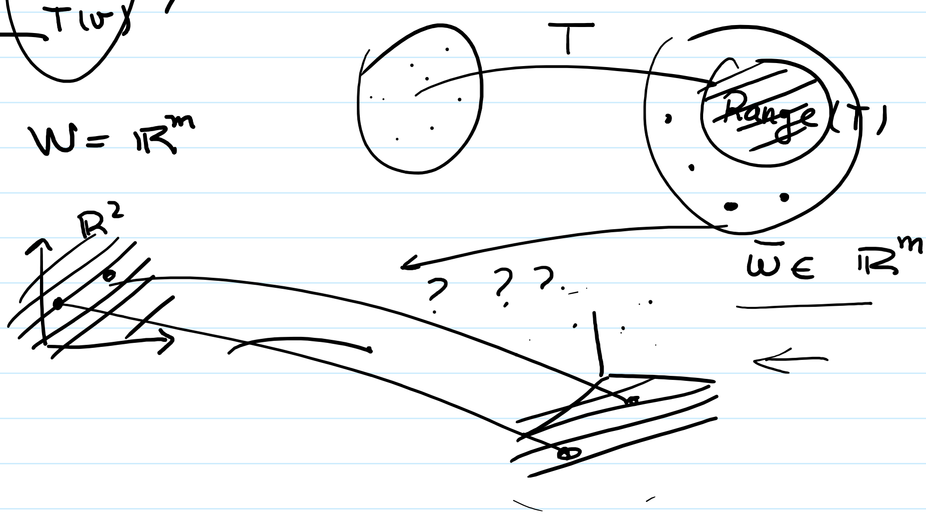
$$A_{m\times n} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \qquad B = \{\vec{e}_1, \dots, \vec{e}_n\}$$

Basis for $\mathbb{R}^n$ domain

1) T 1:1 (injektive)

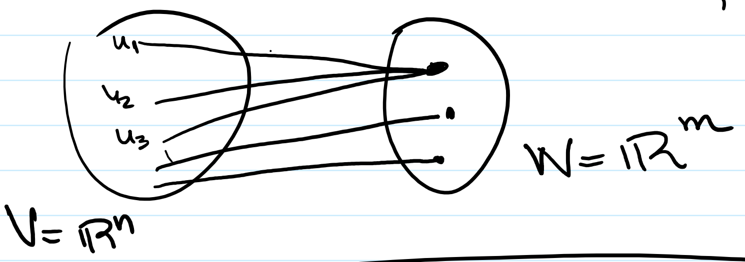
$\vec{u} \neq \vec{v} \Rightarrow T(\vec{u}) \neq T(\vec{v})$





2) T is ONTO-mapping if $\text{Range}(T) = \mathbb{R}^m$ codomain

$\text{Range}(T) = \{ \vec{w} \in \mathbb{R}^m \,/\, \text{there exists at least one vector } \vec{u} \in \mathbb{R}^n,\ T(\vec{u}) = \vec{w} \}$



3) $\boxed{T \text{ is a bijection if } T \text{ is } 1:1 \ \& \ \text{ONTO-mapping.}}$

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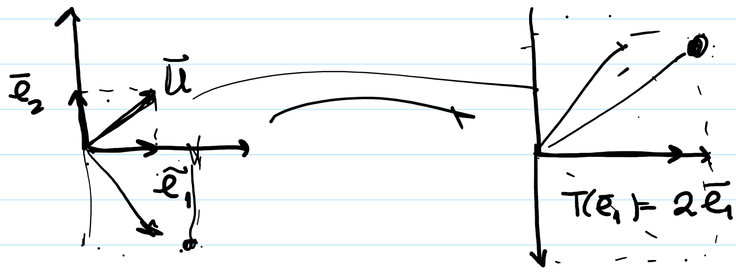
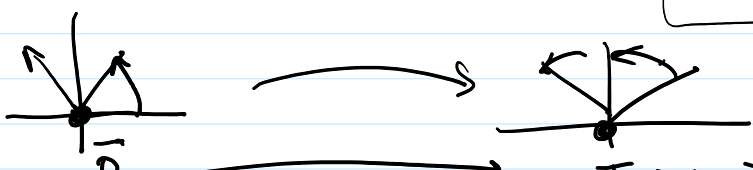


$T(\bar{e}_1) = 2\bar{e}_1 = (2,0)$

$T(\bar{e}_2) = -2\bar{e}_2 = (0,-2)$

$\bar{u} = \alpha \bar{e}_1 + \beta \bar{e}_2$

$T(u) = \alpha T(\bar{e}_1) + \beta T(\bar{e}_2) = \alpha \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -2 \end{pmatrix}$



<u>Theorem 12</u> page 94 Lay § 1.9

Hyp: $T: \mathbb{R}^n \to \mathbb{R}^m$

$u \mapsto T(u) = A_{m\times n} [u_{n\times 1}] = [T(u)]_{m\times 1}$

$B = \{\bar{e}_1, .., \bar{e}_n\}$ Basis for $\mathbb{R}^n$ = domain.

a) T onto-mapping. $\iff$ $\underset{m\times n}{A}\ \underset{n\times 1}{\overset{\in \mathbb{R}^n}{X}} = b_{m\times 1}$ has always a solution for any choice of $b_{m\times 1} \in \mathbb{R}^m$.

$$A_{m\times n} = \begin{bmatrix} \underbrace{T(\bar{e}_1)\ T(\bar{e}_2)}\ \dots\ T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

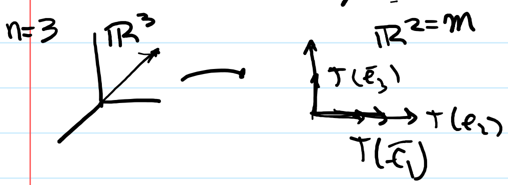
$$x_1 \begin{bmatrix} T(\bar{e}_1) \end{bmatrix} + x_2 \begin{bmatrix} T(\bar{e}_2) \end{bmatrix} + \dots + x_n \begin{bmatrix} T(e_n) \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$b_{m\times 1}$ is a linear combination of $[T(\bar{e}_1)], \dots [T(\bar{e}_n)]$

$$\Longleftrightarrow \quad b_{m\times 1} \in \text{Span}\left\{ [T(\bar{e}_1)], \dots [T(\bar{e}_n)] \right\} = \text{Range}(T)$$

$$\Longleftrightarrow \quad \text{Range}(T) = \mathbb{R}^m.$$

Obs: $n > m$ $\quad$ $\text{Span}\{T(\bar{e}_1), \dots, T(\bar{e}_n)\} =$



$\text{Span}\{\underbrace{T(\bar{e}_{j_1}) \dots T(e_{j_m})}\}$

$\neq m$ elements LI.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

b) T 1:1 $\Longleftrightarrow$ $\{T(\bar{e}_1), T(\bar{e}_2) \dots T(e_n)\}$ LI

$$A_{m\times n} = \begin{bmatrix} Te_1 \dots Te_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$ only the Trivial solution.

c) $\underbrace{T \text{ bijection}}$ $\Longleftrightarrow$ T 1:1 $\quad$ $\underline{Ax = 0}$ has only 1 solution (Trivial $x=0$)

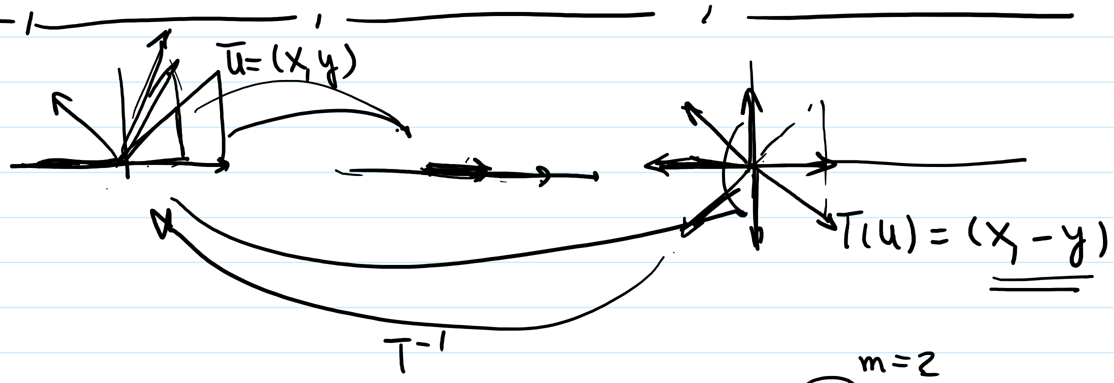
T onto $\quad$ $\underline{Ax = b}$ has always a solutions for all b

and it has to be unique solution

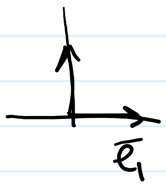
$$x = \underbrace{x_H} + x_p = \vec{0} + x_p$$

$$x = \underbrace{x_H}_{} + x_p = 0 + x_p$$

only one solution.

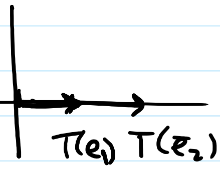


$\mathbb{R}^{m=2}$



$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$T(\bar{e}_1)$ $T(\bar{e}_2)$



$\text{Range}(T) = \mathbb{R}^1$

$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\text{Range} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$

$$= \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

Example:

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = \text{matrix} \simeq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \text{vector}$$

$$= [T(\bar{e}_1)]_{n \times 1} \quad \text{colum.}$$

(a_1, a_2, a_3) vector v

$$v^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{3 \times 1} \text{ matrix}$$

$[T(\bar{e}_1)]$ = colum $\quad (T(\bar{e}_1))$ vector $\quad (a_1, a_2, \dots a_n) = \bar{v}$

T bijection $\Longleftrightarrow \exists\, \underbrace{T^{-1} : \mathbb{R}^m \to \mathbb{R}^n}_{}$ $\quad \underline{n = m}$

T bijection $\iff \exists\ \underbrace{T^{-1} : \mathbb{R}^m \to \mathbb{R}^n}$ $\quad \underline{n=m}$

$T: \mathbb{R}^n \to \mathbb{R}^m \quad T \leftrightarrow A_{m\times n}$ $\quad T^{-1} \leftrightarrow [\underbrace{A^{-1}}]$

* $A x = \vec{0}$ uniq solution

$A x = \vec{b}$ unique solution

$\underline{b \in \text{Span}\{T(\bar{e}_1) \dots T(\bar{e}_n)\}}$

$\{\underbrace{Te_1, \dots T(\bar{e}_n)}\}$ LI

$\{Te_1, \dots T(\bar{e}_n)\}$ is a basis for $\mathbb{R}^n$

Square Matrices.

$$\boxed{A_{n\times n} A^{-1}_{n\times n} = A^{-1}_{n\times n} A_{n\times n} = I_{n\times n}}$$

If T is inversible:

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{T^{-1}} \mathbb{R}^n$$

$$u \longrightarrow \underbrace{T(u)}_{\bar{w}} \quad \text{---} \quad T^{-1}(\bar{w}) = \bar{u}$$

$$\underbrace{T^{-1}}(T(u))$$

$$\mathbb{R}^m \xrightarrow{T^{-1}} \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

$$\bar{w} \qquad \underbrace{T^{-1}((w))}_{u} \qquad T(\bar{u}) = \bar{w}$$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$\bar{e}_1 = (1,0) \longrightarrow T(\bar{e}_1) = (3,5)$

$\bar{e}_2 = (0,1) \longrightarrow T(\bar{e}_2) = (4,6)$

$\underbrace{\bar{u} = (x,y)}$

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}}_{A_{2\times 2}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+4y \\ 5x+6y \end{bmatrix}$$

is this T a bijection?

? what is T^{-1} ?

If $A_{n\times n}$ has inverse A^{-1} so T is inversible

$T^{-1} = [A^{-1}]$

$? \quad A \cdot A^{-1} = I \; ?$

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} \boxed{3} & 4 & 1 & 0 \\ 5 & \boxed{6} & 0 & 1 \end{array}\right]$$ Augmented matrix.

Echelon form $\underbrace{I}$ $\underbrace{\boxed{} \quad \boxed{}}_{A^{-1}}$

Solution $AX = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Solution $AX = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left[\begin{array}{cc|cc} \boxed{3} & 4 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right] \quad L_2 \leftarrow L_2 - \underbrace{\tfrac{5}{3}}L_1$$

$\det A = 18 - 20 =$

$= -2 \neq 0$

$$\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ \cancel{5 - \tfrac{5}{3}3} & 6 - \tfrac{5}{3} \cdot 4 & 0 - \tfrac{5}{3}(1) & 1 - \tfrac{5}{3}(0) \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 0 & -\tfrac{2}{3} \cdot -\tfrac{3}{2} & -5/3 & 1 \end{array}\right] \quad L_2 \leftarrow -\tfrac{3}{2} L_2$$

$$\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 0 & 1 & \tfrac{5}{2} & -3/2 \end{array}\right] \quad L_1 \leftarrow L_1 - 4L_2$$

$$\left[\begin{array}{cc|cc} 3-0 & 4-4(1) & 1 - \cancel{4}^{2}\left(\tfrac{5}{\cancel{2}}\right) & 0 + \cancel{4}^{2}\left(+\tfrac{3}{\cancel{2}}\right) \end{array}\right.$$

$$\left[\begin{array}{cc|cc} 3-0 & 4-4(1) & 1-\cancel{4}^{2}\left(\frac{5}{\cancel{2}}\right) & 0+\cancel{4}^{2}\left(+\frac{3}{\cancel{2}}\right) \\ 0 & 1 & 5/2 & -3/2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} \cancel{3}/3 & 0 & -9/3 & 6/3 \\ 0 & 1 & 5/2 & -3/2 \end{array}\right] \longleftarrow$$

$$\left[\underbrace{\begin{array}{cc} \boxed{1} & 0 \\ 0 & \boxed{1} \end{array}}_{I} \middle| \underbrace{\begin{array}{cc} -3 & 2 \\ 5/2 & -3/2 \end{array}}_{A^{-1}}\right]$$

$$-\frac{18+20}{2} = \frac{2}{2} = 1$$

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}}_{A^{-1}} = \begin{bmatrix} -9+\frac{20}{2} & 6-\cancel{4}\cdot\frac{3}{\cancel{2}} \\ \cancel{-15}+\cancel{6}\cdot\frac{5}{\cancel{2}} & \underbrace{10-\cancel{6}^{3}\cdot\frac{3}{\cancel{2}}}_{1} \end{bmatrix}$$

0

+15

our inverse matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$$

$$\bar{w} \to T^{-1}(w) = \underbrace{\begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}}_{A^{-1}} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} -3a+2b \\ \frac{5}{2}a-\frac{3}{2}b \end{bmatrix}}_{\text{law of } T^{-1}}$$

standard matrix

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x,y) \to T(x,y) = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det A = a \cdot d - b \cdot c \neq 0 \Rightarrow A^{-1} \therefore T^{-1} \text{ inverse}$$

transf

whose matrix

is A^{-1}

Using Cramer's method, we can have a formula for the inverse $A^{-1}_{2\times 2}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_1 = \frac{\det A_1\binom{1}{0}}{\det A} = \frac{\det\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}}{ad - bc} = \frac{d}{\det A}$$

$$x_2 = \frac{\det A_2\binom{1}{0}}{\det A} = \frac{\det\begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix}}{\det A} = \frac{-c}{\det A}$$

$$y_1 = \frac{\det A_1\binom{0}{1}}{\det A} = \frac{\det\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}}{\det A} = \frac{-b}{\det A}$$

$$y_2 = \frac{\det A_2\binom{0}{1}}{\det A} = \frac{\det\begin{bmatrix} a & 0 \\ c & 1 \end{bmatrix}}{\det A} = \frac{a}{\det A}$$

$$\boxed{A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

<u>Def</u>:

$$A_{m\times n} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

$$A^T_{n\times m} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}$$ Transpose matrix.

<u>Theorem</u> 6 page 123 Lay

a) If $A_{n\times n}$ is inversible, Then $A^{-1}_{n\times n}$ is also inversible

a) If $A_{n\times n}$ is invertible, then $A_{n\times n}$ is also invertible

$$\boxed{(A^{-1})^{-1} = A}$$

2) $A_{n\times n}$, $B_{n\times n}$ invertible matrices Then

$(A.B)_{n\times n}$ is also invertible

$$(AB)^{-1} = B^{-1}.A^{-1}$$

3) $A_{n\times n}$ invertible, then A^T is also invertible

$$(A^T)^{-1} = (A^{-1})^T$$

Obs & demonstration ideas.

1) A^{-1} invertible $\Rightarrow \exists D$ such that $\boxed{A^{-1}.D = D.A^{-1} = I}$

Obs $\boxed{D = A} \Rightarrow A^{-1}.A = A.A^{-1} = I$

$$= (A^{-1})^{-1} = A$$

2) $(A.B)(AB)^{-1} = (A(B)(B^{-1})A^{-1}) = (A(I)A^{-1} =$

$= A.A^{-1} = I$

$(AB)^{-1}.(AB)$

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§ 2.2

Final Theoretical Result **Theorem 8** page 130

$A_{n\times n}$: The following eq

1) $A_{n\times n}$ is a inversible matrix $\Longleftrightarrow$ T inversible Transf

b) $A_{n\times n}$ is row equivalent to $I_{n\times n}$ (Echelon form of A is $I_{n\times n}$)

c) A has n pivots positions $[A|X] \longleftrightarrow [I|A^{-1}]$ gaussian elemination

d) $AX=0$ has ONLY the trivial solutions

e) $\{\underbrace{[a_1],[a_2]\dots[a_n]}_{\text{columns from } A}\}$ L I set.

f) $T(x) = A.x$ is injective (1:1)

$Ax=b$ has also 1 unique solution

g) $Ax=b$ has one unique solution fol all choices of $b\in\mathbb{R}^n$

h) $\text{span}\{[a_1],\dots[a_n]\} = \mathbb{R}^n$

i) $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is onto

j) $\boxed{A.A^{-1} = A^{-1}.A = I}$

k) A^T is also inversible.