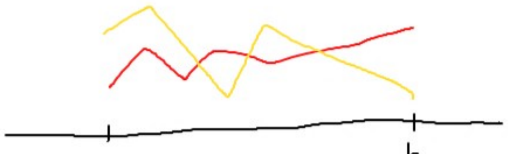
Idag: 9.13 (boken)

$$\begin{cases} \ddot{u} - u'' = 0, & x \in \mathbb{R},\ t \in \mathbb{R}_+ \\ u(x,0) = u_0(x) \\ \dot{u}(x,0) = v_0(x) \end{cases}$$

$u(x,t)$ $\quad \dot{u} = \frac{\partial}{\partial t} u \quad u' = \frac{\partial}{\partial x} u$



Låt $v = \dot{u}$

$$\begin{cases} \dot{u} - v = 0 \\ \dot{v} - u'' = 0 \end{cases} \Rightarrow \begin{cases} \dot{u}\varphi - v\varphi = 0 \\ \dot{v}\varphi - u''\varphi = 0 \end{cases}$$

Vad är Ψ?

$$\Psi_{n-1}(t) = \frac{t_n - t}{k}$$

$$\Psi_n(t) = \frac{t - t_{n-1}}{k}$$

Låt $I_n := (t_{n-1}, t_n]$, väljer en partition av $[0,T] = \bigcup_{n=1}^{N} I_n$, integrera över (a,b)

$$\int_{I_n}\int_a^b \dot{u}\varphi \, dx\,dt - \int_{I_n}\int_a^b v\varphi \, dx\,dt = 0 \quad (1)$$

$$\int_{I_n}\int_a^b \dot{v}\varphi \, dx\,dt - \int_{I_n}\int_a^b u''\varphi \, dx\,dt = 0 \Rightarrow \int_{I_n}\int_a^b \dot{v}\varphi \, dx\,dt + \int_{I_n}\int u'\varphi' \, dx\,dt = 0 \quad (2)$$

Ansätter

$$U(x,t) = U_{n-1}(x)\Psi_{n-1}(t) + U_n(x)\Psi_n(t)$$

$$V(x,t) = U_{n-1}(x)\Psi_{n-1}(t) + V_n(x)\Psi_n(t)$$

Jämför $U(x,t) = \sum_{j=1}^{m} \xi_j(t)\varphi_j(x)$

Nästa steg:

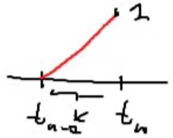
Stoppa in $U(x,t)$, $V(x,t)$

i (1) och (2)

$$\int_{I_n}\int_a^b \frac{U_n - U_{n-1}}{k}\,\varphi_j\,dx\,dt - \int_{I_n}\int_a^b (V_{n-1}\psi_{n-1} + V_n\psi_n)\,\varphi_j\,dx\,dt = 0 \qquad j=1,\dots,m \quad (m = \#\text{punkter i partitionen})$$

$$\int_{I_n}\int_a^b \frac{V_n - V_{n-1}}{k}\,\varphi_j\,dx\,dt + \int_{I_n}\int_a^b (U'_{n-1}\psi_{n-1} + U'_n\psi_n)\,\varphi_j'\,dx\,dt = 0$$

$$\left(\int_{I_n}\psi_{n-1}\,dt = \int_{I_{n-1}}\psi_n\,dt = \frac{k}{2}\right)$$



Iterativ form: "allt med n = allt med $n-1$"

$$\int_a^b (U_n - U_{n-1})\varphi_j\,dx - \frac{k}{2}\int_a^b (V_{n-1} + V_n)\varphi_j\,dx = 0$$

$$\int_a^b (V_n - V_{n-1})\varphi_j\,dx + \frac{k}{2}\int_a^b (U'_{n-1} + U'_n)\varphi_j'\,dx = 0$$

$$\Longrightarrow$$

$$\int_a^b U_n\varphi_j\,dx - \frac{k}{2}\int_a^b V_n\varphi_j\,dx = \frac{k}{2}\int_a^b V_{n-1}\varphi_j\,dx + \int_a^b U_{n-1}\varphi_j\,dx$$

$$\int_a^b V_n\varphi_j\,dx + \frac{k}{2}\int_a^b U_n'\varphi_j'\,dx = \int_a^b V_{n-1}\varphi_j\,dx - \frac{k}{2}\int_a^b U'_{n-1}\varphi_j'\,dx$$

$$\bar{U}_n = \begin{pmatrix}\xi_1\\ \vdots \\ \xi_m\end{pmatrix} \qquad S = \frac{1}{h}\begin{pmatrix} 1 & -1 & & & 0\\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & 2 & \\ 0 & & & \ddots & 1\end{pmatrix}$$

$$\bar{V}_n = \begin{pmatrix}\zeta_1\\ \vdots \\ \zeta_m\end{pmatrix} \qquad M = \frac{h}{6}\begin{pmatrix} 2 & 1 & & & 0\\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ 0 & & 1 & 4 & \\ & & & & \ddots\ 2\end{pmatrix}$$

$$M\bar{U}_n - \frac{k}{2}M\bar{V}_n = \frac{k}{2}M\bar{V}_{n-1} + M\bar{U}_{n-1}$$

$$M\bar{V}_n + \frac{k}{2}S\bar{U}_n = M\bar{V}_{n-1} - \frac{k}{2}S\bar{U}_{n-1}$$

(Vill ha $Ax=b$)

$$\begin{bmatrix} M & -\frac{k}{2}M \\ \frac{k}{2}S & M\end{bmatrix}\begin{bmatrix}\bar{U}_n \\ \bar{V}_n\end{bmatrix} = \begin{bmatrix} M & \frac{k}{2}M \\ -\frac{k}{2}S & M\end{bmatrix}\begin{bmatrix}\bar{U}_{n-1} \\ \bar{V}_{n-1}\end{bmatrix}$$