TMA683 HT 2020
David Cohen (david.cohen@chalmers.se) Chalmers & GU

Introduction to convolutions

December 3, 2020

This document is based on an earlier version of Fardin Saedpanah from 2019.

A *convolution* is a mathematical operation taking two functions as input and producing a third function as output. Convolutions have interesting applications in connection with Laplace transforms. Most importantly, convolutions are used in image processing (edge detection, smoothing of images, gimp, photoshop), in electrical engineering, or in artificial intelligence for instance.

Definition. *The convolution of two functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined by (the new function)*

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau)\, d\tau$$

if the integral is bounded.

In the context of Laplace transforms, we assume that f and g are causal. In this case, we may reduce the limits of integration to where the integrand is non-zero, and we get the equation

$$(f * g)(t) = \theta(t) \int_{0}^{t} f(t-\tau) g(\tau)\, \mathrm{d}\tau,$$

where we recall that $\theta(t)$ is the Heaviside step function. Thus $(f * g)(t) = 0$ for $t < 0$.

Example. Compute the convolution between $f(t) = t$ and $g(t) = t^2$:

Def conv

$$(f * g)(t) = \int_0^t (t-\tau)\tau^2\, \mathrm{d}\tau = \int_0^t (t\tau^2 - \tau^3)\, \mathrm{d}\tau = t\frac{t^3}{3} - \frac{t^4}{4} = \frac{t^4}{12}.$$

$f(t) = t$
$g(t) = t^2$

Convolutions have the following properties (for example)

Theorem.

a) *The convolution is a bi-linear operation, i. e. for all $\alpha, \beta \in \mathbb{R}$ and all function f, g, h, one has*

$$(\alpha f + \beta g) * h = \alpha (f * h) + \beta (g * h)$$

and similar in the second argument (whenever all terms are well defined).

The convolution also satisfies

b) $f * g = g * f$.

c) $(f * g) * h = f * (g * h) = f * g * h$.

Proof. This is a direct consequence of the definition of convolution. For instance, for b), one has

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,\mathrm{d}\tau = \left\{\begin{matrix} \eta = t-\tau \\ \mathrm{d}\eta = -\mathrm{d}\tau \end{matrix}\right\} = -\int_{\infty}^{-\infty} f(\eta)g(t-\eta)\,\mathrm{d}\eta$$
$$= \int_{-\infty}^{\infty} f(\eta)g(t-\eta)\,\mathrm{d}\eta = (g*f)(t).$$

□

One of the main reason for using convolutions (in this lecture) is their simple Laplace transform. Namely, one has the following result

Theorem (Convolution theorem)**.** *If f and g are causal, with Laplace transforms denoted by $\mathscr{L}\{f\}=F$ and $\mathscr{L}\{g\}=G$. If there exist constants M and a such that $|f(t)|,|g(t)| \leq M\mathrm{e}^{at}$, then one has*

$$\mathscr{L}\{f*g\}(s) = F(s)G(s).$$

In addition, one has

$$\mathscr{L}^{-1}\{F(s)G(s)\}(t) = (f*g)(t).$$

This means that the inverse Laplace transform of a product is a convolution.

!

Proof.

$$\mathscr{L}\{f*g\}(s) = \int_0^{\infty} \mathrm{e}^{-st}\left(\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t = \int_0^{\infty}\int_0^t \mathrm{e}^{-st} f(t-\tau)g(\tau)\,\mathrm{d}\tau\,\mathrm{d}t$$
$$= \left\{\begin{matrix}\text{switch order of integration,} \\ \text{see Figure 1}\end{matrix}\right\} = \int_0^{\infty} g(\tau)\left(\int_{\tau}^{\infty} \mathrm{e}^{-st} f(t-\tau)\,\mathrm{d}t\right)\mathrm{d}\tau$$
$$= \int_0^{\infty} \mathrm{e}^{-s\tau} g(\tau)\left(\int_{\tau}^{\infty} \mathrm{e}^{-s(t-\tau)} f(t-\tau)\,\mathrm{d}t\right)\mathrm{d}\tau$$
$$= \{\text{set } r = t-\tau \text{ in the inner integral}\}$$
$$= \int_0^{\infty} \mathrm{e}^{-s\tau} g(\tau)\left(\int_0^{\infty} \mathrm{e}^{-sr} f(r)\,\mathrm{d}r\right)\mathrm{d}\tau = F(s)G(s).$$

□

Example. Let g be (causal) piecewise continuous and of exponential order. Use the Laplace transform to find the exact solution to the IVP

$$\begin{cases} y''(t) + y(t) = g(t) \\ y(0) = 0, y'(0) = 0. \end{cases}$$

Let us first set $Y(s) = \mathscr{L}\{y\}(s)$ and $G(s) = \mathscr{L}\{g\}(s)$. Taking the Laplace transform of the above differential equation, one then gets (using properties of Laplace transforms)

$$s^2Y(s) + Y(s) = G(s)$$

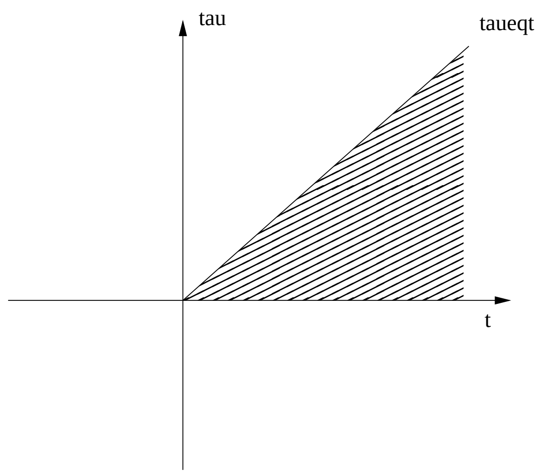

Figure 1: The integration area in the proof of the convolution theorem.

$s^2 Y(s) + Y(s) = G(s)$

or (since $\mathscr{L}\{\sin\}(s) = \frac{1}{s^2+1}$)

$$\underline{Y(s)} = \frac{1}{\underline{s^2+1}} G(s) = \mathscr{L}\{\sin\}(s) \cdot \mathscr{L}\{g\}(s).$$

product of LT!!

That is, $Y(s)$ is given as the product of two Laplace transforms and we can use the above convolution theorem! Taking the inverse Laplace transform, one finally find the solution to the IVP

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}(t) = \underline{\left(\sin * g\right)}(t) \underset{\text{Def}}{=} \int_0^t \sin(t-\tau) g(\tau)\,\mathrm{d}\tau.$$