Chapter 9: Fourier analysis (summary)

December 11, 2020

Goal: Study approximation of functions by simple trigonometric functions.
Applications: Signal processing, .mp3, .jpeg, etc.

- A function $f\colon \mathbb{R} \to \mathbb{C}$ such that there exists a $p > 0$ with $f(x + p) = f(x)$ for all $x \in \mathbb{R}$ is called p-periodic. The smallest such p is called the (prime) period of f.
- For an integrable p-periodic function f, the integral
$$\int_a^{a+p} f(x)\,\mathrm{d}x$$
does not depend on the point a.
- Let $f\colon \mathbb{R} \to \mathbb{C}$ be 2π-periodic and Riemann integrable on $[-\pi, \pi]$. The series
$$\sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx} \qquad \text{or} \qquad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big)$$
are called the Fourier series of f (FS), where
$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x \quad \text{for} \quad n \in \mathbb{Z},$$
and
$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \qquad a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \qquad b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \quad \text{for} \quad n \geq 1,$$
are called the Fourier coefficients of f. One has the relations
$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = \mathrm{i}(c_n - c_{-n}) \quad \text{and} \quad c_n = \frac{a_n - \mathrm{i}b_n}{2}, \quad c_{-n} = \bar{c}_n.$$
Observe that $b_n = 0$ if f is even (i. e. $f(-x) = f(x)$ for all x) and $a_n = 0$ if f is odd (i. e. $f(-x) = -f(x)$ for all x).

  Observe also that one can integrate over any interval of length 2π since f (and cosine and sine) is 2π-periodic.
- The set $\{\mathrm{e}^{\mathrm{i}nx}\}_{n\in\mathbb{Z}}$ is an orthogonal set on $[-\pi, \pi]$, that is
$$\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}nx}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \begin{cases} 0 & \text{if} \quad n \neq k \\ 2\pi & \text{else.} \end{cases}$$
- Bessel's inequality reads: Let f be 2π-periodic and square integrable, then
$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,\mathrm{d}x.$$
In particular, this implies that the Fourier coefficients c_n go to zero as n goes to $\pm$ infinity (Riemann-Lebesgue lemma). A similar formula exists for the coefficients a_n and b_n (see compendium) which implies that a_n, b_n go to zero as n goes to infinity. These facts are needed, for example, to prove convergence results on the Fourier series (see below).

- We recall the following definitions. A function $f\colon [a,b]\to\mathbb{R}$ is piecewise continuous (notation $f\in PC([a,b])$) if f is continuous on $[a,b]$ except perhaps at finitely many points $x_1,x_2,\ldots,x_n\in[a,b]$. At these points the left-hand and right-hand limits of f exist: $f(x_j-)=\lim_{h\to 0,h>0}f(x_j-h)$ and $f(x_j+)=\lim_{h\to 0,h>0}f(x_j+h)$. Similarly, f is piecewise smooth (notation $f\in PS([a,b])$) if $f,f'\in PC([a,b])$. Finally, $f\in PC(\mathbb{R})$, resp. $f\in PS(\mathbb{R})$, if f is piecewise continuous, resp. smooth, on every bounded interval $[a,b]$.

- Pointwise convergence of Fourier series: Consider f a 2π-periodic function and piecewise smooth on $\mathbb{R}$ (i. e. in $PS(\mathbb{R})$). Set $S_N^f(x):=\sum_{n=-N}^{N}c_n\mathrm{e}^{\mathrm{i}nx}$, where c_n are the Fourier coefficients of f. We have
$$\lim_{N\to\infty}S_N^f(x)=\frac{1}{2}\big(f(x-)+f(x+)\big)\quad\forall x\in\mathbb{R}.$$
In particular, if f is continuous at x, $\lim_{N\to\infty}S_N^f(x)=f(x)$ and we see that the Fourier series converges, in this case, to the value of $f(x)$!

- Parseval's identity reads: For $f\in\mathscr{L}^2(-\pi,\pi)$ a piecewise smooth 2π-periodic function, one has
$$\sum_{n=-\infty}^{\infty}|c_n|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,\mathrm{d}x.$$

- Derivative of Fourier series: Let f be a 2π-periodic, continuous and in $PS([-\pi,\pi])$, we have
$$c_n'=\mathrm{i}nc_n,$$
where c_n are the Fourier coefficients of f and c_n' those of f'. In terms of a_n and b_n, one has the relations $a_n'=nb_n$ and $b_n'=-na_n$.
With this in hand, one has the following result:
Let f be 2π-periodic, continuous, and piecewise smooth and suppose that f' is piecewise smooth. If $\sum_{n=-\infty}^{\infty}c_n\mathrm{e}^{\mathrm{i}nx}$ is the Fourier series of $f(x)$, then $f'(x)$ is the derived series $\sum_{n=-\infty}^{\infty}\mathrm{i}nc_n\mathrm{e}^{\mathrm{i}nx}$ for all x at which $f'(x)$ exists. At jump points of f', the series converges to $\frac{1}{2}\big(f'(x-)+f'(x+)\big)$.

- Integral of Fourier series: Let f be 2π-periodic and in $PC(\mathbb{R})$ with Fourier coefficients c_n. Set $F(x)=\int_0^x f(y)\,\mathrm{d}y$. If $c_0=0$ then the Fourier coefficients of F are given by
$$C_n=\frac{c_n}{\mathrm{i}n}\quad\text{for}\quad n\neq 0$$
and $C_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}F(x)\,\mathrm{d}x$. I. e. $F(x)=C_0+\sum_{n\neq 0}\frac{c_n}{\mathrm{i}n}\mathrm{e}^{\mathrm{i}nx}$. If $c_0\neq 0$, this series converges to $F(x)-c_0x$. (This comes from the fact that the integral of a periodic function may not be periodic: $f(x)=1$ is periodic but its integral $F(x)=x$ is not).

We now look at Fourier series of functions of arbitrary period.

- Using a simple change of variable, the Fourier series of a $2L$-periodic function f is given by
$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}\Big(a_n\cos\Big(\frac{n\pi}{L}x\Big)+b_n\sin\Big(\frac{n\pi}{L}x\Big)\Big)$$
with the Fourier coefficients
$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\Big(\frac{n\pi}{L}x\Big)\,\mathrm{d}x\quad\text{and}\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\Big(\frac{n\pi}{L}x\Big)\,\mathrm{d}x.$$

We now look at Fourier series of non-necessary periodic functions.

- Let f be defined on the interval $[0,\pi]$ and integrable. Using the even extension of f on $[-\pi,\pi]$ defined by

  $$f_{\text{even}}(-x) = f(x) \quad \text{for} \quad x \in [0,\pi] \quad \text{(observe that} \quad f_{\text{even}}(x) = f(x) \text{ for } x \in [0,\pi])$$

  one gets the Fourier cosine series of f

  $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

  with the coefficients

  $$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) \, \mathrm{d}x.$$

  Using the odd extension of f on $[-\pi,\pi]$ defined by

  $$f_{\text{odd}}(-x) = -f(x) \quad \text{for} \quad x \in (0,\pi) \quad \text{and} \quad f_{\text{odd}}(0) = 0$$

  one gets the Fourier sine series of f

  $$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

  with the coefficients

  $$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, \mathrm{d}x.$$

Further resources:

- wikiversity.org
- wikibooks.org
- math.lamar.edu
- mathsisfun.com
- khanacademy.org
- intmath.com