

Chapter 10: Partial differential equations (summary)

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Goal: Present a tool to find exact solutions to particular PDEs.

- Consider partial differential equations of the form

$$\mathcal{L}(u) = F,$$

where \mathcal{L} is a **linear differential operator** (i. e. $\mathcal{L}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{L}(u_1) + c_2 \mathcal{L}(u_2)$ for any constants c_1, c_2) and $F = F(x)$ can be regarded as an external force for example. This abstract equation is called **homogeneous** if $F \equiv 0$ and **inhomogeneous** if $F \neq 0$. Furthermore, one has to add, to the above problem, **linear boundary conditions** of the form

$$B(u) = f \quad \text{on the boundary,}$$

where B is again a linear operator and $f = f(x)$. We have **homogeneous boundary conditions** if $f \equiv 0$, **inhomogeneous boundary conditions** otherwise.

Example: The abstract form of the one-dimensional linear inhomogeneous heat equation

$$\begin{cases} u_t(x, t) = 3u_{xx}(x, t) + x, \\ u(0, t) = u(1, t) = 0, \\ u(x, 0) = \sin(x) \end{cases}$$

is given by taking $\mathcal{L}(u) = u_t - 3u_{xx}$, $F(x) = x$, and $B(u) = u$, $f = 0$ and initial condition $u(x, 0) = \sin(x)$.

- The **superposition principle** can be used to solve smaller or easier problems from a more complicated original problem. We have the following: If u_1, u_2, \dots, u_k satisfy the linear PDEs

$$\begin{aligned} \mathcal{L}(u_j) &= F_j \\ B(u_j) &= f_j \quad j = 1, 2, \dots, k \end{aligned}$$

and c_1, c_2, \dots, c_k are any constants, then $u := c_1 u_1 + c_2 u_2 + \dots + c_k u_k$ satisfies the more complicated problem

$$\begin{aligned} \mathcal{L}(u) &= c_1 F_1 + c_2 F_2 + \dots + c_k F_k \\ B(u) &= c_1 f_1 + c_2 f_2 + \dots + c_k f_k. \end{aligned}$$

The above tells us, for instance, that to solve the inhomogeneous PDE $\mathcal{L}(u) = F, B(u) = f$ it is enough to find the solution, denoted u_H , to the homogeneous PDE $\mathcal{L}(u) = 0, B(u) = 0$ and one particular solution, denoted u_p , to the problem $\mathcal{L}(u) = F, B(u) = f$. The general solution will then be given by $u = u_H + u_p$.

- The technique of **separation of variables** can be used to solve homogeneous linear partial differential equations $\mathcal{L}(u) = 0, B(u) = 0$ with initial conditions if needed. The main idea is to set $u(x, t) = X(x)T(t)$ and find two ordinary differential equations for the functions X and T . If one can solve these differential equations (using in addition the given boundary conditions), one finds solutions to the original PDE (here, one can use Fourier series and the superposition principle).

Observe that this technique works only for particular problems. As application, we have used separation of variables to find exact solutions to the homogeneous heat and wave equations.

Combining the above with the superposition principle, we were also able to find the exact solution to some inhomogeneous heat equation.

Further resources:

- wikipedia.org
- wikibooks.org
- ocw.mit.edu
- ocw.mit.edu
- math.etsu.edu
- www.ucl.ac.uk