(1) Exam may contain more problems than before. Problems of the same type/difficulty than before?

I have not seen all old exams.

(2) Variational formulation: How do I know when to set up both test <u>and</u> trial space?

One <u>always</u> has a test and trial space!

(VF) Find $u \in V$ s.t. .... $\forall v \in \hat{V}$

$V$: trial space (solution to DE lives in)

$\hat{V}$: test space ("to kill" some terms to make sense of integrals)

$\hat{V} = \{ v : [0,1] \to \mathbb{R} : v, v' \in L^2, \; v(0) = v(1) = 0 \}$

(3) What are we expected to know about Euler + C-N?

You have your notes. Be able to apply them.

⚠ $y_n \approx y(t_n)$

$y_n$: numerical solution (can compute)

$y_n \neq y(t_n)$

$y(t_n)$: exact sol. of DE (unknown!!)

④ How do you compute sums using Fourier series?

We have seen this using Parseval identity.
(relates $\sum_{n=1}^{\infty}$ of Fourier coeff. with integrals)

⑤ Info exam: "derivation of the formulas for Fourier coefficients"

Chapter IX. Section 3)

- Uses $\int_{-\pi}^{\pi} e^{inx} e^{-ikx} dx = \begin{cases} 0 & n \neq k \\ 2\pi & n = k \end{cases} \quad \forall n,k \in \mathbb{Z}$

- $f(x) \underset{=}{\sim} \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \Big| \cdot e^{-ikx}$ and $\int_{-\pi}^{\pi}$

$$\int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} c_n e^{inx} e^{-ikx} dx$$

$$= \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} e^{inx} e^{-ikx} dx}_{\substack{0 \text{ if } n \neq k \\ 2\pi \text{ else}}}$$

$$= 2\pi c_k \quad \text{or}$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

① Upp. 6 tentan 2019-04-24

VF + FE + linear system for $\begin{cases} -u''(x) + 6u(x) = -2 \\ u(0) = u'(1) = 0 \end{cases}$

Partition $T_h$ of $[0,1]$ with $h = \frac{1}{2}$.

a) VF: Consider $V = \{ v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1),\ v(0) = 0 \}$

Multiply DE with a test function v and integrate

$$-\int_0^1 u''(x)v(x)dx + 6\int_0^1 u(x)v(x)\,dx = -2\int_0^1 v(x)\,dx \quad \forall v \in \ldots$$

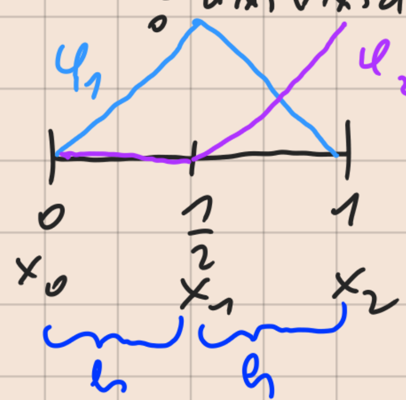
by part

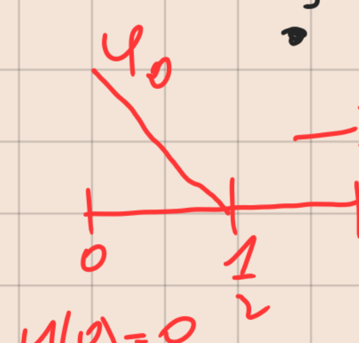
$$-u'(x)v(x)\Big|_0^1 + \int_0^1 u'(x)v'(x)dx = \underbrace{-\underset{0}{u'(1)}v(1) + u'(0)\underset{0}{v(0)}}_{=0\ \rightarrow \text{ simplify problem}} + \int_0^1 u'v'$$

The VF reads: Find $u \in V$, with $u'(1) = 0$, s.t.

$$\int_0^1 u'(x)v'(x)dx + 6\int_0^1 u(x)v(x)\,dx = -2\int_0^1 v(x)dx \quad \forall v \in V.$$

b) FE: $T_h$ :




→ no need of $\varphi_0$

$(\varphi_0(0) = 1)$

$u(0) = 0$

$V_h = \{ v : [0,1] \to \mathbb{R} : v$ cont. piecewise linear on $T_h$, $v(0) = 0\}$

$= \text{Span}(\varphi_1, \varphi_2)$ [ 2 elements ]

The FE problem then reads:

Find $U \in V_h$ s.t. $\int_0^1 U'(x) \chi'(x)\, dx + 6 \int_0^1 U(x) \chi(x)\, dx =$

$= -2 \int_0^1 \chi(x)\, dx \quad \forall \chi \in V_h$

c) To get a linear system from the above, set

$U(x) = \zeta_1 \varphi_1(x) + \zeta_2 \varphi_2(x)$ and test with $\varphi_1, \varphi_2$:

FE solution

unknown coeff.

$V_h = \text{span}(\varphi_1, \varphi_2)$

Insert in FE problem to get:

$$\sum_{k=1}^{2} \zeta_k \int_0^1 \varphi_k'(x) \varphi_j'(x)\, dx + 6 \sum_{k=1}^{2} \zeta_k \int_0^1 \varphi_k(x) \varphi_j(x)\, dx =$$

$$= -2 \int_0^1 \varphi_j(x)\, dx \quad j = 1,2$$

$\underbrace{(S + 6M)}_{A} \vec{\zeta} = \vec{b}$, where

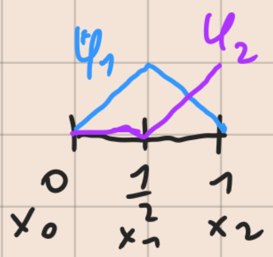
$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1/2 \end{pmatrix} \rightsquigarrow$ solve lin. syst. to get $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$

Stiffness matrix $S$: $s_{kj} = \int_0^1 \varphi_k'(x)\varphi_j'(x)\,dx$

Mass matrix $M$: $m_{kj} = \int_0^1 \varphi_k(x)\varphi_j(x)\,dx$

Last vector $b$: $b_i = -2\int_0^1 \varphi_i(x)\,dx$ for $i = 1,2$

Recall



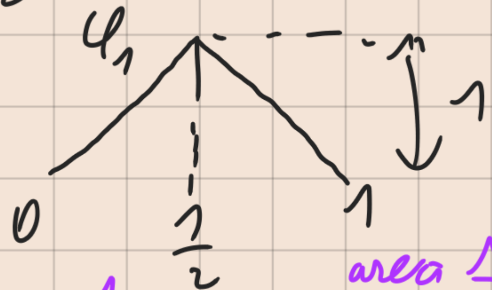

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h} & x_{j-1} \le x \le x_j \\ \frac{x - x_{j+1}}{-h} & x_j \le x \le x_{j+1} \\ 0 & \text{else} \end{cases} \qquad j = 1, \dots, m$$

$$b_1 = -2\int_0^1 \varphi_1(x)\,dx \underset{\text{area }\Delta}{=} -2 \cdot \frac{1}{2} \cdot 1 = -1$$

$\downarrow \frac{1}{2}$

$$b_2 = -2\int_0^1 \varphi_2(x)\,dx = -\frac{1}{2}$$

Area $\Delta$: $\frac{1}{2} \times \overset{1}{\text{Basis}} \times \overset{1}{\text{height}}$




$$m_{11} = \int_0^1 \varphi_1(x)\varphi_1(x)\,dx = \int_0^1 \varphi_1(x)^2\,dx \overset{\text{area }\Delta}{=} \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} \quad (*)$$

$$m_{12} = \int_0^1 \varphi_1(x)\varphi_2(x)\,dx = m_{21}, \quad m_{22} = \int_0^1 \varphi_2^2(x)\,dx$$

④ Tenta 2020.08.20:

**3.** (a) För vilka värden på $a > 0$ är funktionerna $\sin(ax)$ och $\cos(ax)$ ortogonala i $L_2(0,1)$? (4p)
(b) Bestäm Fourier cosine-serien med perioden $2\pi$ till funktionen $f(x) = \sin x$, $0 \le x \le \pi$. (5p)

b) Use def Fourier cosine series:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{, where}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx \quad \text{for } n = 0, 1, 2, \ldots$$

We now compute $a_0$:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} \sin(x)\, dx = \ldots = \frac{4}{\pi}$$

Next, we compute $a_n$ for $n = 1, 2, 3, \ldots$:

trigonometric formula

$$a_n \overset{\text{Def } f}{\underset{b}{=}} \frac{2}{\pi} \int_0^{\pi} \sin(x)\cos(nx)\, dx \overset{b}{=}$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin((1-n)x)\, dx + \frac{1}{\pi} \int_0^{\pi} \sin((1+n)x)\, dx$$

$$-\frac{\cos((1-n)x)}{(1-n)} \Big|_0^{\pi} \qquad n \neq 1 \qquad (**)$$

**4.** Betrakta den inhomogena vågekvationen (10p)

$$\begin{cases} \ddot{u}(x,t) - u''(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \ u(1,t) = 1, & t > 0, \\ u(x,0) = 2x & 0 < x < 1 \\ \dot{u}(x,0) = 0 & 0 < x < 1 \end{cases}$$

Använd variabelseparationsmetoden för att bestämma $u(x,t)$.

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,t) = 0, \ \boxed{u(1,t) = 1} \\ u(x,0) = 2x \\ u_t(x,0) = 0 \end{cases}$$

inhomogeneous wave eq.

(i) Ansatz: $u(x,t) = v(x,t) + s(x)$ and find 2 easier problems for $s(x)$ and $v(x,t)$

$s(x) \to$ BVP, $v(x,t) \to$ homog. PDE

(ii) Insert ansatz into PDE.

$$\begin{cases} v_{tt} - v_{xx} - s'' = 0 \\ v(0,t) + s(0) = 0, \ v(1,t) + s(1) = 1 + 0 \\ v(x,0) + s(x) = 2x \\ v_t(x,0) = 0 \end{cases}$$

(iii) We then get:

$$\begin{cases} -s''(x) = 0 \\ s(0) = 0, \ s(1) = 1 \end{cases}$$

BVP

$\Downarrow$

$s(x)$

$$\begin{cases} v_{tt} - v_{xx} = 0 \\ v(0,t) = 0, \ v(1,t) = 0 \\ v(x,0) = 2x - s(x) \\ v_t(x,0) = 0 \end{cases}$$

hom. wave eq.

**5.** Härled *variationsformulering* för begynnelsevärdesproblemet ($a, b, \alpha, \beta$ är icke-nollkonstanter),

$$\begin{cases} -u'' + au' + bu = f, \quad 0 < x < 1, \\ u'(0) = \alpha, \quad u(1) = \beta. \end{cases}$$

Set $a = 1 = b$ for simplicity.

1) In order to find the VF, we multiply DE with test function $v$ and integrate:

$$\underbrace{-\int_0^1 u''(x)v(x)\,dx}_{\text{by part}} + \int_0^1 u'(x)v(x)\,dx + \int_0^1 u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

$$\underline{-u'(1)v(1)} + \underset{\alpha}{\underset{\shortparallel}{u'(0)}}v(0) + \int_0^1 u'(x)v'(x)\,dx$$

(inhomogeneous Neumann BC)

We <u>need</u> this term, so we don't "kill it".

→ We set $v(1) = 0$ in order for this term to die.

• This motivates the following choices:

Test space $\tilde{V} = \{ v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1) \text{ and } v(1) = 0 \}$

Trial space $V = \{ v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1) \text{ and } v(1) = \beta \}$

Because of non-hom. Dirichlet BC

• The VF then reads:

Find $u \in V$ s.t.

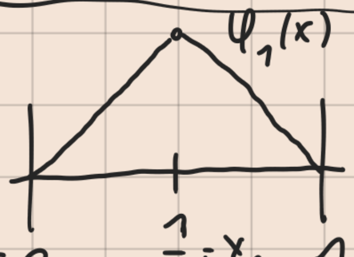
$$\alpha \cdot u(0) + \int_0^1 u(x)v(x)\,dx + \int_0^1 u'(x)v(x)\,dx + \int_0^1 u(x)v'(x)\,dx$$

$$= \int_0^1 f(x)v(x)\,dx - \alpha \cdot v(0) \quad \forall v \in \tilde{V}.$$

(*) $$m_{11} = \int_0^1 \varphi_1(x)^2\,dx = \int_0^{1/2} (2x)^2\,dx + \int_{1/2}^1 (2(x-1))^2\,dx =$$

$$= 4\frac{x^3}{3}\Big|_0^{1/2} + 4\left(\frac{x-1}{3}\right)^3\Big|_{1/2}^1 =$$

$$= \frac{4}{3}\left(\frac{1}{8} + \frac{1}{8}\right) = \frac{1}{3}$$




$h_1 = \frac{1}{2}$

$$\varphi_1(x) = \begin{cases} \dfrac{x - x_0}{h_1} & x_0 \le x \le x_1 \\ \dfrac{x - x_2}{-h_1} & x_1 \le x \le x_2 \end{cases} = \begin{cases} 2x & \text{for } 0 \le x \le \frac{1}{2} \\ -2(x-1) & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$$

(**) For $n \neq 1$: $a_n = \frac{1}{\pi} \int_0^{\pi} \sin((1-n)x)\,dx +$

$$+ \frac{1}{\pi} \int_0^{\pi} \sin((1+n)x)\,dx = \ldots$$

$$= \frac{2\left((-1)^{n+1} - 1\right)}{\pi\,(n^2-1)} \quad \text{for } n \neq 1$$

For $n=1$: $a_1 = \frac{2}{\pi} \int_0^{\pi} \sin(x)\cos(x)\,dx = 0$

This gives us the Fourier cosine series

$$f(x) \sim \underbrace{\frac{2}{\pi}}_{\frac{a_0}{2}} + \frac{2}{\pi} \sum_{n=2}^{\infty} \underbrace{\frac{\left((-1)^{n+1} - 1\right)}{n^2-1}}_{a_n} \cos(nx) =$$

$$= \frac{2}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-2)}{4k^2-1} \cos(2kx)$$

$(-1)^{n+1} - 1 = 0$ for $n$ odd

$= -2$ for $n$ even $\rightsquigarrow n = 2k$