Chapter 3: Polynomial approximations in $1d$ (summary)

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Goal: We want to approximate (complicated) functions by (easy) polynomials. As an application, we shall use this to find numerical approximations to solutions to differential equations (introduced in this chapter and further studied in later chapters).

- For a positive integer N, the space of trigonometric polynomials on $[0,L]$ is defined as

$$T^N(0,L) = \text{span}\left(1, \cos(\frac{2\pi}{L}x), \sin(\frac{2\pi}{L}x), \cos(\frac{2\pi}{L}2x), \sin(\frac{2\pi}{L}2x), \dots, \cos(\frac{2\pi}{L}Nx), \sin(\frac{2\pi}{L}Nx)\right)$$
$$= \left\{ f(x) = \sum_{n=0}^{N} \left(a_n \cos(\frac{2\pi}{L}nx) + b_n \sin(\frac{2\pi}{L}nx)\right) \,:\, a_n, b_n \in \mathbb{R} \right\}.$$

  We shall study this functions space later in the lecture.

- Consider an interval $[a,b]$ and a grid of $(q+1)$ distinct points $x_0 = a < x_1 < \ldots < x_q = b$. One defines Lagrange polynomials by

$$\lambda_i(x) = \prod_{j=0, j\neq i}^{q} \frac{x - x_j}{x_i - x_j}$$

  for $i = 0, 1, \dots, q$. One then has (no proof)

$$\mathscr{P}^{(q)}(a,b) = \text{span}\left(\lambda_0(x), \lambda_1(x), \dots, \lambda_q(x)\right).$$

- Denote a partition of the interval $[0,1]$ into $m+1$ subintervals by $\tau_h : 0 = x_0 < x_1 < \ldots < x_m < x_{m+1} = 1$, where $h_j = x_j - x_{j-1}$ for $j = 1, 2, \dots, m+1$. We define the hat function $\{\varphi_j\}_{j=0}^{m+1}$ by

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h_j} & \text{for } x_{j-1} \le x \le x_j \\ \frac{x - x_{j+1}}{-h_{j+1}} & \text{for } x_j \le x \le x_{j+1} \\ 0 & \text{else} \end{cases}$$

  for $j = 1, \dots, m$. The functions $\varphi_0(x)$ and $\varphi_{m+1}(x)$ are defined as half hat functions.

  With the above, one then defines the space of continuous piecewise linear functions on $[0,1]$ by

$$V_h = V_h(0,1) = \left\{ v\colon [0,1] \to \mathbb{R} \quad : \; v \text{ cont. piecewise linear on } \tau_h \right\} = \text{span}\left(\varphi_0, \varphi_1, \dots, \varphi_{m+1}\right).$$

  As usual, one has $v(x) = \sum_{j=0}^{m+1} \zeta_j \varphi_j(x)$, where $\zeta_j = v(x_j)$, for any $v \in V_h$.

- For a positive integer q and $f \in L^2(a,b)$, one defines its L^2-projection as the polynomials $Pf \in \mathscr{P}^{(q)}(a,b)$ verifying

$$\int_a^b f(x)p(x)\,\mathrm{d}x = \int_a^b (Pf)(x)p(x)\,\mathrm{d}x \quad \text{for all} \quad p \in \mathscr{P}^{(q)}(a,b)$$

  or shortly

$$(f,p)_{L^2(a,b)} = (Pf,p)_{L^2(a,b)} \quad \text{for all} \quad p \in \mathscr{P}^{(q)}(a,b)$$

or (since monomials x^j are basis of $\mathscr{P}^{(q)}(a,b)$)

$$(f,x^j)_{L^2(a,b)} = (Pf,x^j)_{L^2(a,b)} \quad \text{for} \quad j=0,1,\dots,q.$$

Theoretical results: The L^2-projection Pf is unique and the best approximation of f in $\mathscr{P}^{(q)}(a,b)$ in the L^2-norm.

- In a nutshell, a Galerkin finite element method (FEM) for the BVP with homogeneous Dirichlet BC

$$\begin{cases} -u''(x) = f(x) \quad \text{for} \quad x\in(0,1) \\ u(0)=0, u(1)=0 \end{cases}$$

consists of the following

1. Multiply the DE by a test function $v \in V^0 = \{v\colon [0,1]\to\mathbb{R}:\ v,v'\in L^2(0,1) \text{ and } v(0)=v(1)=0\}$.
2. Integrate the above over the domain $[0,1]$ and get the variational formulation of the problem (VF)

$$\text{Find} \quad u\in V^0 \quad \text{such that} \quad \int_0^1 u'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \quad \text{for all} \quad v\in V^0$$

or shortly

$$\text{Find} \quad u\in V^0 \quad \text{such that} \quad (u',v')_{L^2(0,1)} = (f,v)_{L^2(0,1)} \quad \forall v\in V^0.$$

3. Specify the finite dimensional space $V_h^0 \subset V^0$ defined as $V_h^0 = \operatorname{span}(\varphi_1,\dots,\varphi_m)$, for the above hat functions φ_j. Consider the FE problem

$$\text{Find} \quad U\in V_h^0 \quad \text{such that} \quad (U',V')_{L^2(0,1)} = (f,V)_{L^2(0,1)} \quad \forall V\in V_h^0.$$

4. Insert the ansatz

$$U(x) = \sum_{j=1}^m \zeta_j\varphi_j(x)$$

into the FE problem and take $V=\varphi_i$, for $i=1,\dots,m$, to get a linear system of equation for the unknown $\zeta=(\zeta_1,\dots,\zeta_m)$:

$$A\zeta = b.$$

Here, A is termed the stiffness matrix (with entries $a_{ij} = (\varphi_i',\varphi_j')_{L^2(0,1)}$) and b the load vector (with entries $b_i = (f,\varphi_i)_{L^2(0,1)}$).

Further resources:

- https://www.youtube.com/watch?v=GtJKUIG9KXI&ab_channel=OscarVeliz
- https://web.stanford.edu/class/energy281/FiniteElementMethod.pdf
- http://mitran-lab.amath.unc.edu/courses/MATH762/bibliography/LinTextBook/chap6.pdf
- https://www.youtube.com/watch?v=WwgrAH-IMOk&ab_channel=SeriousScience (good!)