1) Crank-Nicolson scheme:

Prob: $\begin{cases} \dot{y}(t) = f(y(t)) \\ y(0) = y_0 \end{cases}$ $y_0 \in \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$ given, $y(t) = ?$ unknown, $0 \leq t \leq T$

For a stepsize $k \left(= \frac{T}{N}\right)$, the C-N scheme reads

$$y_{\ell+1} = y_\ell + k \underbrace{\left(\frac{f(y_\ell) + f(y_{\ell+1})}{2} \right)}_{\text{average of } f/\text{right hand side on } [t_\ell, t_{\ell+1}].}$$

and gives approx. $y_\ell \approx y(t_\ell)$

Application: $\dot{y}(t) + S' y(t) = F(t)$, where S' matrix, $F(t)$ vector

$\Leftrightarrow \quad \dot{y}(t) = F(t) - S' y(t)$

CN reads: $y_{\ell+1} = y_\ell + \frac{k}{2}\left((F(t_\ell) - S' y_\ell) + (F(t_{\ell+1}) - S' y_{\ell+1}) \right)$

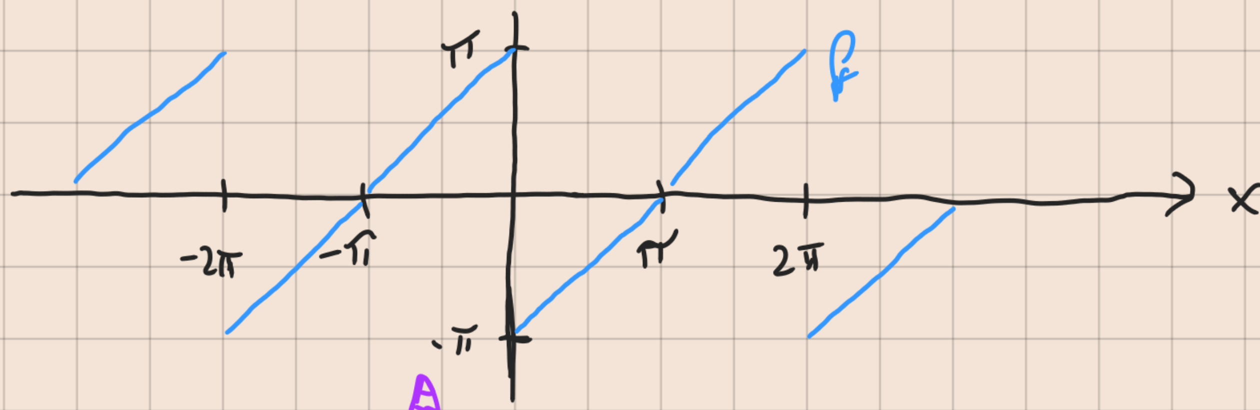
or $\left(I + \frac{k}{2} S' \right) y_{\ell+1} = \left(I - \frac{k}{2} S' \right) y_\ell + \frac{k}{2} \left(F(t_\ell) + F(t_{\ell+1}) \right)$

linear system $A y_{\ell+1} = b \rightsquigarrow$ solve it and get $y_{\ell+1}$.

2) Problem 17, lecture notes:

$f(x)$ odd, 2π-periodic, where $f(x) = x - \pi$ for $0 < x \leq \pi$

a) Graph $f(x)$ for $-3\pi \leq x \leq 3\pi$?



to make f odd

b) FS of $f(x)$?

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

f odd $\Rightarrow a_n \equiv 0 \ \forall n = 0, 1, 2, \ldots$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx =$$

Def b_n Def $f(x)$, see a)

$$= \frac{1}{\pi} \int_{-\pi}^{0} (\pi + x) \sin(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} (x - \pi) \sin(nx)\, dx =$$

$$= \frac{1}{\pi} \left[\frac{\sin(nx) - n(x+\pi)\cos(nx)}{n^2} \right]_{x=-\pi}^{0} + \frac{1}{\pi} \left[\frac{\sin(nx) + n(\pi - x)\cos(nx)}{n^2} \right]_{x=0}^{\pi}$$

$$= \frac{\sin(n\pi) - n\pi}{n^2 \pi} + \frac{\sin(n\pi) - n\pi}{n^2 \pi} = -\frac{2n\pi}{n^2 \pi} = -\frac{2}{n}$$

c) ... Parseval should give the answer

3) Computation of load vector?

In general, one has $\vec{b} = (b_i)_{i=1}^{m}$, where

$b_i = \int_0^1 f(x)\varphi_i(x)\,dx$ for $i = 1, \dots, m$, where

φ_i is the hat function : $\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & x_{i-1} \le x \le x_i \\ \dfrac{x - x_{i+1}}{-h} & x_i \le x \le x_{i+1} \\ 0 & \text{else} \end{cases}$

(I do not do the half hats)

Hence, $b_i = \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{h}\right) f(x)\,dx + \int_{x_i}^{x_{i+1}} \left(\frac{x - x_{i+1}}{-h}\right) f(x)\,dx$

$$= \frac{1}{h} \underbrace{\int_{x_{i-1}}^{x_i} (x - x_{i-1}) f(x)\,dx}_{} - \frac{1}{h} \int_{x_i}^{x_{i+1}} (x - x_{i+1}) f(x)\,dx$$

same

if "easy to integrate" ok
else quadrature formula
(depends on f)

4) Computation of stiffness matrix $S = (s_{ij})_{i,j=1}^{m}$?

By def $s_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)\,dx$, where

[I don't look at half hat]

Diagonal elements:

$$s_{ii} = \int_0^1 (\varphi_i'(x))^2 dx = \int_{x_{i-1}}^{x_{i+1}} (\varphi_i'(x))^2 dx = \int_{x_{i-1}}^{x_i} (\varphi_i'(x))^2 dx + \int_{x_i}^{x_{i+1}} (\varphi_i'(x)) dx$$

Def $\varphi_i(x)$

$$= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)^2 dx = \frac{h}{h^2} + \frac{h}{h^2} = \frac{2}{h}$$

Def $h = x_i - x_{i-1} = x_{i+1} - x_i$

Above diagonal

Def φ_i

$$s_{i,i+1} = \int_0^1 \varphi_i'(x)\varphi_{i+1}'(x)dx = \int_{x_i}^{x_{i+1}} \varphi_i'(x)\cdot\varphi_{i+1}'(x)dx = -\frac{1}{h^2}\int_{x_i}^{x_{i+1}} dx =$$

Def h

$$= -\frac{1}{h^2}\cdot h = -\frac{1}{h}$$

$s_{i-1,i}$ same, $s_{ij} = 0$ if $|i-j| > 2$.

5) Ex 2.1b in exercises.pdf:

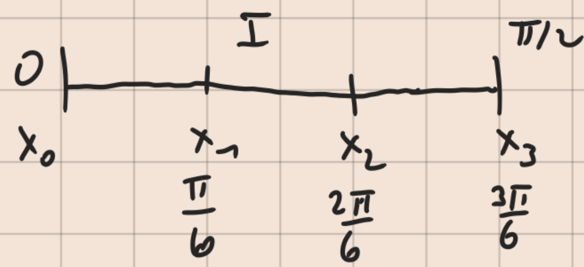
Find pw linear interpolant $\pi_h f(x)$ on I which is divided in 3 intervals of same length

Compute error

$$\|\pi_h v - v\|_{L^p} \le C\cdot h^2 \|v''\|_{L^p}$$

One has $f(x) = \sin(x)$ and $I = [0, \frac{\pi}{2}]$

Hence



hence $h = \frac{\pi}{6}$ and $m = 2$

Def $\Pi_h f$: $\quad \Pi_h f(x) = \sum_{j=0}^{m+1} f(x_j)\varphi_j(x)$, with hat fct φ_j

Hence $\Pi_h f(x) = f(0)\varphi_0(x) + f(\frac{\pi}{6})\varphi_1(x) + f(\frac{\pi}{3})\varphi_2(x) + f(\frac{\pi}{2})\varphi_3(x)$

$$\overset{\text{Def } f}{=} 0 + \frac{1}{2}\varphi_1(x) + \frac{\sqrt{3}}{2}\varphi_2(x) + \varphi_3(x) =$$

$$\overset{\text{facit}}{=} \begin{cases} \frac{3}{\pi}x & \text{for } 0 \le x < \pi/6 \quad (*) \\ \frac{3}{\pi}(\sqrt{3}-1)x + 1 - \frac{\sqrt{3}}{2} & \text{for } \pi/6 \le x < \pi/3 \\ \frac{3}{\pi}(2-\sqrt{3})x + \frac{3\sqrt{3}}{2} - 2 & \text{for } \pi/3 \le x < \pi/2 \end{cases}$$

(*) If $0 \le x < \pi/6$: $\Pi_h f(x) \underset{\text{Def } \varphi_j(x)}{=} \frac{1}{2}\varphi_1(x) \underset{(1)}{=} \frac{1}{2}\left(\frac{x - x_0}{h}\right) = \frac{1}{2}\frac{x}{h} \overset{\text{Def } h}{=} \frac{6}{2}\frac{x}{\pi}$

(1): Def $\varphi_1(x)$ on $x_0 = 0 \le x \le \pi/6 = x_1$

Error in L^1 f.ex:

Use $\|\Pi_h f - f\|_{L^1(0,\pi/2)} \le C \cdot h^2 \cdot \|f''\|_{L^1(0,\pi/2)}$, where

$$\|f''\|_{L^1(0,\pi/2)} \underset{\uparrow}{=} \int_0^{\pi/2} |(\sin(x))''| dx = \int_0^{\pi/2} |-\sin(x)| dx = -\cos(x)\Big|_{x=0}^{\pi/2} =$$

Def norm, f

$$= 1$$

$$\hookrightarrow \|\Pi_h f - f\|_{L^1(0,\pi/2)} \leq C \cdot h^2 \cdot 1 \underset{\uparrow}{=} C \cdot \frac{\pi^2}{36}$$

Def h

6) Solve $\begin{cases} y''(t) + y'(t) + y(t) = e^{t/2} \\ y(0) = 0, \; y'(0) = 1 \end{cases}$ with LT ?

- Take LT: $s^2 Y(s) - s y(0) - y'(0) + s Y(s) - y(0) + Y(s) = \dfrac{2}{2s-1}$,

(table)

where $Y(s) = \mathcal{L}\{y\}(s)$.

- Solve for $Y(s)$: $Y(s) = \dfrac{2s+1}{2s-1} \cdot \dfrac{1}{(s^2+s+1)}$.

Write $Y(s) = \dfrac{A}{2s-1} + \dfrac{Bs+C}{s^2+s+1} = \dfrac{As^2 + As + A + 2Bs^2 - Bs - C + 2sC}{(2s-1)(s^2+s+1)}$.

This gives $2s+1 \overset{!}{=} As^2 + As + A + 2Bs^2 - Bs - C + 2sC$

or $A + 2B = 0, \; A - B + 2C = 2, \; A - C = 1$

Solving for A, B, C gives $A = \frac{8}{7}$, $B = -\frac{4}{7}$, $C = \frac{1}{7}$.

Hence:

$$Y(s) = \frac{8}{7}\,\frac{1}{2s-1} - \frac{1}{7}\,\frac{4s-1}{(s+1/2)^2+3/4}$$

• Take ILT:

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = \frac{8}{7}\,\mathcal{L}^{-1}\left\{\frac{1}{2s-1}\right\}(t) - \frac{1}{7}\,\mathcal{L}^{-1}\left\{\frac{4s}{(s+1/2)^2+3/4}\right\}(t)$$

$$+ \frac{1}{7}\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1/2)^2+3/4}\right\}(t) \quad =$$

Table

$$= \frac{8}{7}\cdot\frac{1}{2}\cdot e^{t/2} - \frac{1}{7}\left(-\frac{4}{\sqrt{3}}\,e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) + 4e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$$+ \frac{1}{7}\left(\frac{2}{\sqrt{3}}\,e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right) = \ldots$$

ouf...