Chapter 2: Mathematical tools (summary)

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Goal: Introduce some (abstract) spaces and various mathematical tools. This will help us to solve (numerically) differential equations in the next chapters.

- A set V is called a vector space or linear space (VS) if, for all $u, v, w \in V$ and for all $\alpha, \beta \in \mathbb{R}$ one has

  1. $u + \alpha v \in V$ (linearity)
  2. $(u + v) + w = u + (v + w) = u + v + w$ (associativity)
  3. There exists an element $0 \in V$ such that $u + 0 = 0 + u = u$ for all $u \in V$ (identity element)
  4. For all $u \in V$, there exists an element $(-u) \in V$ such that $u + (-u) = 0$ (inverse element)
  5. $u + v = v + u$ (commutativity)
  6. $(\alpha + \beta)u = \alpha u + \beta u$
  7. $\alpha(u + v) = \alpha u + \beta v$
  8. $\alpha(\beta u) = (\alpha\beta)u = \alpha\beta u$
  9. There exists $1 \in \mathbb{R}$ such that $1u = u$ for all $u \in V$.

  (Technical comment: the last condition is not written in the book, but should be part of the definition if one wants to write it clearly).

  The elements in V are called vectors (but they can be something else, like "normal" vectors, matrices, functions, or sequences) and the ones in $\mathbb{R}$ scalars. The above axioms (rules) tell us that we can do anything reasonable with vectors and scalars.

  Example: The vector space of all polynomials, defined on $\mathbb{R}$, of degree $\leq n$ is denoted by

  $$\mathscr{P}^{(n)}(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : \ a_0, a_1, \ldots, a_n \in \mathbb{R}\}.$$

- A subset $U \subset V$ of a VS V is called a subspace of V if $\alpha u + \beta v \in U$ for all $u, v \in U$ and $\alpha, \beta \in \mathbb{R}$.

- Let V be a VS. The space of all linear combinations of the elements $v_1, v_2, \ldots, v_n \in V$ is denoted by

  $$\mathrm{span}(v_1, \ldots, v_n) = \{a_1 v_1 + a_2 v_2 + \ldots + a_n v_n : \ a_1, \ldots, a_n \in \mathbb{R}\}.$$

  Example: $\mathrm{span}(1, x, x^2) = \{a_0 1 + a_1 x + a_2 x^2 : \ a_0, a_1, a_2 \in \mathbb{R}\} = \mathscr{P}^{(2)}(\mathbb{R})$.

- A set $\{v_1, v_2, \ldots, v_n\}$ in a VS V is linearly independent if the equation

  $$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0 \in V$$

  has only the trivial solution $a_1 = a_2 = \ldots = a_n = 0 \in \mathbb{R}$. Else it is called linearly dependent.

  Example: The set $\{1, x, x^2\} \in \mathscr{P}^{(2)}(\mathbb{R})$ is linearly independent.

- A set $\{v_1, v_2, \ldots, v_n\}$ in a VS V is called a basis of V if the set is linearly independent and $\mathrm{span}(v_1, \ldots, v_n) = V$. The dimension of V is then given by the number of elements of this set, here $\dim(V) = n$.

  Example: The set $\{1, x, x^2\}$ is a basis of $\mathscr{P}^{(2)}(\mathbb{R})$ and thus $\dim(\mathscr{P}^{(2)}(\mathbb{R})) = 2$.

- A scalar product or inner product on a VS V is a map $(\cdot,\cdot)\colon V\times V\to\mathbb{R}$ such that, for all $u,v,w\in V$ and $\alpha\in\mathbb{R}$,

  1. $(u,v)=(v,u)$ (symmetry)
  2. $(u+\alpha v,w)=(u,w)+\alpha(v,w)$ (linearity)
  3. $(u,u)\geq 0$ (positivity)
  4. $(u,u)=0\in\mathbb{R}$ if and only if $u=0\in V$.

- A VS V with an inner product is called an inner product space, which is denoted by $(V,(\cdot,\cdot))$ or $(V,(\cdot,\cdot)_V)$ or $(V,\langle\cdot,\cdot\rangle_V)$.

  Such space has a norm defined by $\|v\|=\sqrt{(v,v)}$ for all $v\in V$.

  Example: The space of square integrable functions defined on the interval $[a,b]$ is denoted by

  $$L^2([a,b])=L^2(a,b)=L_2(a,b)=\{f\colon[a,b]\to\mathbb{R}\colon \int_a^b |f(x)|^2\,\mathrm{d}x<\infty\}.$$

  It is equipped with the inner product

  $$(f,g)_{L^2}=\int_a^b f(x)g(x)\,\mathrm{d}x$$

  which induces the norm

  $$\|f\|_{L^2}=\sqrt{(f,f)_{L^2}}=\sqrt{\int_a^b |f(x)|^2\,\mathrm{d}x}.$$

- Let $(V,(\cdot,\cdot))$ be an inner product space and $u,v\in V$. u and v are orthogonal if $(u,v)=0$. Notation: $u\perp v$.

- Let $(V,(\cdot,\cdot))$ be an inner product space and $u,v\in V$. Cauchy–Schwarz inequality (CS) reads

  $$|(u,v)|\leq\|u\|\cdot\|v\|.$$

- Let $(V,(\cdot,\cdot))$ be an inner product space and $u,v\in V$. The triangle inequality ($\triangle$) reads

  $$\|u+v\|\leq\|u\|+\|v\|.$$

- The space of continuous function defined on $[a,b]$ is given by

  $$C^0([a,b])=\mathscr{C}^0([a,b])=\mathscr{C}^{(0)}(a,b)=\{f\colon[a,b]\to\mathbb{R}:f\text{ is continuous}\}$$

  and equipped with the norm

  $$\|f\|_{C^0([a,b])}=\max_{a\leq x\leq b}|f(x)|.$$

  (Technical observation: one should have a suppremum sup *in place of* max*, see further resources if you are interested. We will try to avoid this technicality in the lecture, hopefully.)*

  Similarly, one can define the space of continuously differentiable functions

  $$C^1([a,b])=\mathscr{C}^1([a,b])=\mathscr{C}^{(1)}(a,b)=\{f\colon[a,b]\to\mathbb{R}:f,f'\text{ are continuous}\}$$

  and equipped with the norm

  $$\|f\|_{C^1([a,b])}=\|f\|_{C^0([a,b])}+\|f'\|_{C^0([a,b])}=\max_{a\leq x\leq b}(|f(x)|+|f'(x)|).$$

  And the space $C^k([a,b])$ of k time continuously differentiable functions ....

- For $1 \leq p < \infty$, we consider the spaces

$$L^p([a,b]) = L_p(a,b) = \{f\colon [a,b] \to \mathbb{R} : \|f\|_{L^p} < \infty\},$$

with the L^p-norm

$$\|f\|_{L^p} = \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

For $''p = \infty`$, one has

$$L^\infty([a,b]) = L_\infty(a,b) = \{f\colon [a,b] \to \mathbb{R} : \|f\|_{L^\infty} < \infty\},$$

with the L^∞-norm

$$\|f\|_{L^\infty} = \max_{a \leq x \leq b} |f(x)|.$$

(Technical observation: one should have an ess.sup in place of the maximum. But you can (probably) forget this comment. We will stay in the easiest situations.)

Further resources:

- https://sv.wikipedia.org/wiki/Linj%C3%A4rt_rum
- https://sv.wikipedia.org/wiki/Inre_produktrum
- https://sv.wikipedia.org/wiki/Lp-rum
- https://sv.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_olikhet
- https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk09/09_basis_and_dimension.html
- https://brilliant.org/wiki/bases/
- https://web.auburn.edu/holmerr/2660/Textbook/innerproduct-print.pdf
- https://terrytao.files.wordpress.com/2008/03/function_spaces1.pdf