TMA683 Tillämpad matematik
Övningsuppgifter (boken FEM)

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This document contains the exercises from the compendium from M. Asadzadeh (23.08.2018).

1. Chapter 4

4.1 Prove that $V_0^{(q)} = \{v \in \mathcal{P}^{(q)}(0,1), v(0) = 0\}$ is a subspace of $\mathcal{P}^{(q)}(0,1)$.

4.3 Consider the ODE

$$\dot{u}(t) = u(t), \quad 0 < t < 1, \quad u(0) = 1.$$

Compute its Galerkin approximation in $\mathcal{P}^{(q)}(0,1)$ for $q = 1, 2, 3, 4$.

4.4 Compute the stiffness matrix and load vector in a finite element approximation of the BVP

$$-u''(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0$$

with $f(x) = x$ and $h = 1/4$.

4.5 We want to find a solution approximation $U(x)$ to

$$-u''(x) = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

using the ansatz $U(x) = A\sin(\pi x) + B\sin(2\pi x)$.

(a) Calculate the exact solution $u(x)$.

(b) Write down the residual $R(x) = -U''(x) - 1$.

(c) Use the orthogonality condition

$$\int_0^1 R(x)\sin(n\pi x)\,\mathrm{d}x = 0, n = 1, 2$$

to determine the constants A and B.

(d) Plot the error $e(x) = |u(x) - U(x)|$.

4.6 Consider the BVP

$$-u''(x) + u(x) = x, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

(a) Verify that the exact solution to the above problem reads

$$u(x) = x - \frac{\sinh(x)}{\sinh(1)}.$$

(b) Let $U(x)$ be a solution approximation defined by

$$U(x) = A\sin(\pi x) + B\sin(2\pi x) + C\sin(3\pi x),$$

where A, B, C are unknown constants. Compute the residual

$$R(x) = -U''(x) + U(x) - x.$$

(c) Use the orthogonality conditions

$$\int_0^1 R(x)\sin(n\pi x)\,\mathrm{d}x = 0, n = 1, 2, 3$$

to determine the constants A, B, C.

4.7 Let $U(x) = \zeta_0\phi_0(x) + \zeta_1\phi_1(x)$ be a solution approximation to

$$-u''(x) = x - 1, \quad 0 < x < \pi, \quad u'(0) = u(\pi) = 0,$$

where ζ_0 and ζ_1 are unknown coefficients and $\phi_0(x) = \cos(\frac{x}{2})$, $\phi_1(x) = \cos(\frac{3x}{2})$.

(a) Find the analytical solution $u(x)$.

(b) Define the residual $R(x)$.

(c) Compute the constants ζ_0 and ζ_1 using the orthogonality conditions

$$\int_0^\pi R(x)\phi_i(x)\,\mathrm{d}x = 0, i = 0, 1.$$

I.e. by projecting $R(x)$ onto the vector space spanned by $\phi_0(x)$ and $\phi_1(x)$.

4.8 Use the projection technique of the previous exercise to solve

$$-u''(x) = 0, \quad 0 < x < \pi, \quad u(0) = 0, u(\pi) = 2,$$

with $U(x) = A\sin(x) + B\sin(2x) + C\sin(3x) + \frac{2}{\pi^2}x^2$ and using the test functions $\{\sin(x), \sin(2x), \sin(3x)\}$.

2. Chapter 5

5.1 Consider two real numbers $a < b$. By defintion of Lagranges polynomials, one has

$$\lambda_a(x) = \frac{b - x}{b - a} \quad \text{and} \quad \lambda_b(x) = \frac{x - a}{b - a}.$$

Show that

$$\lambda_a(x) + \lambda_b(x) = 1 \quad \text{and} \quad a\lambda_a(x) + b\lambda_b(x) = x.$$

Give a geometric interpretation by plotting $\lambda_a(x), \lambda_b(x), \lambda_a(x)+\lambda_b(x)$ and $a\lambda_a(x), b\lambda_b(x), a\lambda_a(x)+b\lambda_b(x)$.

5.2 Consider the following functions defined for $x \in [0, 1]$:

$$f(x) = x^2 \quad \text{and} \quad g(x) = \sin(\pi x).$$

Find their linear interpolants, denoted by $\Pi f \in \mathcal{P}(0, 1)$, resp. $\Pi g \in \mathcal{P}(0, 1)$. In the same figure, plot f and Πf, as well as g and Πg.

5.3 Determine the linear interpolant of the function, defined for $x \in [-\pi, \pi]$,

$$f(x) = \frac{1}{\pi^2}(x-\pi)^2 - \cos^2(x - \frac{\pi}{2}),$$

where the interval $[-\pi, \pi]$ is divided into 4 equal subintervals.

5.15 Prove that

$$\int_{x_0}^{x_1} f'(\frac{x_0+x_1}{2})(x - \frac{x_0+x_1}{2})\,\mathrm{d}x = 0.$$

5.16 Prove that

$$\left|\int_{x_0}^{x_1} f(x)\,\mathrm{d}x - f(\frac{x_0+x_1}{2})(x_1-x_0)\right| \leq \frac{1}{2}\max_{[x_0,x_1]}|f''(x)|\int_{x_0}^{x_1}(x - \frac{x_0+x_1}{2})^2\,\mathrm{d}x$$
$$\leq \frac{1}{24}(x_1-x_0)^3\max_{[x_0,x_1]}|f''(x)|.$$

Hint: Use a Taylor expansion of f about $x = \frac{x_0+x_1}{2}$.

3. Chapter 7

7.1 Consider the two-point BVP

$$-u''(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

Let $V = \{v\colon \|v\| + \|v'\| < \infty, v(0) = v(1) = 0\}$ where $\|\cdot\|$ denotes the L_2-norm.

(a) Use V to derive a variational formulation for the above BVP.

(b) Discuss why V is valid as a vector space of test functions.

(c) Classify which of the following functions are admissible test functions:

$$\sin(\pi x), \quad x^2, \quad x\ln(x), \quad \mathrm{e}^x - 1, \quad x(1-x).$$

7.3 Consider the two-point BVP

$$-u''(x) = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

Let $\mathcal{T}_h : x_j = \frac{j}{4}, j = 0,1,2,3,4$ denote a partition of the interval $0 < x < 1$ into four subintervals of equal length $h = 1/4$. Let V_h be the corresponding space of continuous piecewise liner functions vanishing at $x = 0$ and $x = 1$.

(a) Compute a finite element approximation $U \in V_h$ to the above BVP.

(b) Prove that $U \in V_h$ is unique.

7.5 Consider the two-point BVP, for $x \in I = (0,1)$:

$$-(a(x)u'(x))' = f(x)$$
$$u(0) = 0, \quad a(1)u'(1) = g_1,$$

where a is a positive function and g_1 a constant.

(a) Derive the variational formulation of the above problem.

(b) Discuss how the boundary conditions are implemented.

7.6 Consider the two-point BVP, for $x \in I = (0, 1)$,

$$-u''(x) = 0$$
$$u(0) = 0, u'(1) = 7.$$

Divide the interval I into two subintervals of length $h = \frac{1}{2}$. Let V_h be the corresponding space of continuous piecewise linear functions vanishing at $x = 0$.

(a) Formulate a finite element method for the above problem.

(b) Calculate by hand the finite element approximation $U \in V_h$ to the above BVP.

(c) Study how the boundary condition at $x = 1$ is approximated.

7.7 Consider the two-point BVP

$$-u''(x) = 0, \quad 0 < x < 1, \quad u'(0) = 5, u(1) = 0.$$

Let $\mathcal{T}_h : x_j = \frac{j}{N}, j = 0, 1, \ldots, N, h = 1/N$ denote a uniform partition of the interval $0 < x < 1$ into N subintervals. Let V_h be the corresponding space of continuous piecewise linear functions.

(a) Use V_h, with $N = 3$, and formulate a finite element method for the above problem.

(b) Compute the finite element approximation $U \in V_h$ assuming $N = 3$.

7.8 Consider the problem of finding a solution approximation to

$$-u''(x) = 1, \quad 0 < x < 1, \quad u'(0) = u'(1) = 0.$$

Let $\mathcal{T}_h$ be a partition of the interval $0 < x < 1$ into two subintervals of equal length $h = \frac{1}{2}$. Let V_h be the corresponding space of continuous piecewise linear functions.

(a) Can you find an exact solution to the above problem by integrating twice?

(b) Compute a finite element approximation $U \in V_h$ to u if possible.

7.11 Consider the finite element method applied to

$$-u''(x) = 0, \quad 0 < x < 1, \quad u(0) = \alpha, u'(1) = \beta,$$

where α and β are given constants. Assume that the interval $[0, 1]$ is divided into three subintervals of equal length $h = 1/3$ and that $\{\varphi_j\}_{j=0}^3$ is a nodal basis of V_h, the corresponding space of continuous piecewise linear functions.

(a) Verify that the ansatz

$$U(x) = \alpha\varphi_0(x) + \zeta_1\varphi_1(x) + \zeta_2\varphi_2(x) + \zeta_3\varphi_3(x),$$

yields the following system of equations

$$\frac{1}{h}\begin{pmatrix}-1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix}=\begin{pmatrix}0\\0\\ \beta\end{pmatrix}. \tag{1}$$

(b) If $\alpha = 2$ and $\beta = 3$ show that (1) can be reduced to

$$\frac{1}{h}\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix}=\begin{pmatrix}2h^{-1}\\ 0\\ 3\end{pmatrix}.$$

(c) Solve the above system of equation to find $U(x)$.

7.13 Consider the following eigenvalue problem

$$-au''(x)+bu(x)=0,\quad 0\le x\le 1,\quad u(0)=u'(1)=0,$$

where $a, b > 0$ are constants. Let $\mathcal{T}_h : 0 = x_0 < x_1 < \ldots < x_N = 1$, be a non-uniform partition of the interval $0 \le x \le 1$ into N intervals of length $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$. Let V_h be the corresponding space of continuous piecewise linear functions. Compute the stiffness and mass matrices.

7.14 Show that the FEM with mesh size h for the problem

$$\begin{cases}-u''(x)=1 & 0<x<1\\ u(0)=7, u'(1)=0, & \end{cases}$$

with $U(x) = 7\varphi_0(x)+U_1\varphi_1(x)+\ldots+U_m\varphi_m(x)$ leads to the linear system of equations $\tilde{A}\tilde{U} = \tilde{b}$, where $\tilde{A}\in\mathbb{R}^{m\times(m+1)}$, $\tilde{U}\in\mathbb{R}^{(m+1)\times 1}$, $\tilde{b}\in\mathbb{R}^{m\times 1}$ are given by

$$\tilde{A}=\frac{1}{h}\begin{pmatrix}-1 & 2 & -1 & 0 & \ldots & 0\\ 0 & -1 & 2 & -1 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & 0 & \ldots & 0\end{pmatrix}, \tilde{U}=\begin{pmatrix}7\\ U_1\\ \vdots\\ U_m\end{pmatrix}, \tilde{b}=\begin{pmatrix}h\\ \vdots\\ h\\ h/2\end{pmatrix}.$$

The above reduces to $AU = b$, with

$$A=\frac{1}{h}\begin{pmatrix}2 & -1 & 0 & \ldots & 0\\ -1 & 2 & -1 & \ldots & 0\\ \ldots & \ldots & -1 & 2 & -1\\ 0 & 0 & \ldots & -1 & 2\end{pmatrix}, U=\begin{pmatrix}U_1\\ \vdots\\ U_m\end{pmatrix}, b=\begin{pmatrix}h+\frac{7}{h}\\ \vdots\\ h\\ h/2\end{pmatrix}.$$

4. Chapter 8

8.5a) Compute the solution of

$$\dot{u}(t) + a(t)u(t) = t^2, \quad 0 < t < T, \quad u(0) = 1,$$

where $a(t) = 4$.

5. Chapter 9

9.7 Consider the inhomogeneous problem

$$\begin{cases} u_t(x,t) - \varepsilon u_{xx}(x,t) = f(x,t), \quad 0 < x < 1, t > 0 \\ u(0,t) = u_x(1,t) = 0, \quad t > 0 \\ u(x,0) = u_0(x), \quad 0 < x < 1. \end{cases}$$

Show that for the corresponding stationary problem, $u_t = 0$, one has

$$\|u_x\| \leq \frac{1}{\varepsilon} \|f\|.$$

9.13 Consider the wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, \quad x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x), \quad x \in \mathbb{R} \\ u_t(x,0) = v_0(x), \quad x \in \mathbb{R}. \end{cases}$$

Plot the graph of $u(x,2)$ in the following cases:

(a) $v_0 = 0$ and

$$u_0(x) = \begin{cases} 1, \quad x < 0 \\ 0, \quad x > 0. \end{cases}$$

(b) $u_0 = 0$ and

$$v_0(x) = \begin{cases} -1, \quad -1 < x < 0 \\ 1, \quad 0 < x < 1 \\ 0, \quad |x| > 1. \end{cases}$$