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String theory (FFM485/FIM480(GU))
Lecture notes on 2020/2021 MSc course

Lecturer: Bengt E.W. Nilsson, Physics Department, Chalmers

email: tfebn@chalmers.se
office: Origo north wing, 6th floor, room xxxxx
mobile: 0704-101283

Abstract: This course is based on the textbook *A first course in String Theory* (2nd Ed. Cambridge 2009) by Barton Zwiebach from MIT, Boston. The book has two parts:
Part 1 is called *Basics* and starts from very basic physical principles and provides a logical chain of arguments that ends with a clear picture of our present understanding of string theory. This part, plus Chapter 24, of the book will be covered in the lectures almost completely. The Canvas lecture notes will have some additional comments and details on the superstring which is the most important kind of string theory. These notes will also have some material on M-theory which is a theory that unifies all the different string theories into a unique theory living in 11 spacetime dimensions. This first part of the course is accompanied by 15 rather short home problems, that is, one problem per chapter of the book. The plan is to cover Part 1 (plus Chap 24) in the first five weeks of the course. This means reading over 300 pages but some material in the beginning should be familiar to Swedish students. Some less important sections of Part 1 of the book will also be identified in the lectures.
Part 2 of the book is called *Developments*. It contains separate chapters on a number of absolutely crucial features of string theory that any serious course must discuss. This part of the course contains Chapters 15 - 21, and 23. (Chapter 24 from Part 2 is included in the lectures for Part 1.) These chapters are only summarised in the lectures during week 6 and 7 of the course. This way the course (having in total 16 lectures) leaves time for a final short project that the student can choose from either one of the chapters summarised in Part 2 or from a list of slightly more advanced problems provided at the end of the course. The short project will also require a short report and a short presentation of the results in a seminar (15 minutes plus 5 for questions on zoom).

Examination consists of three parts:
1. 14 home problems (3 points each, 20 points are needed to pass).
2. A short project with report and presentation (on zoom and graded).
3. A mandatory oral exam (45 min on zoom).
Final grade: 1. and 2. have together weight 1/3, and 3. has weight 2/3.

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1 Lecture 1

Read BZ, chapters 1 and 2 and these lecture notes. Then do the first home problem (Hp):

Hp 1: Deadline: November 13, 2020, at noon sharp.
Read sections 1 and 2 of the paper Mike Duff on Kaluza-Klein theory, hep-th/9410046, and answer the following questions. No calculations are required![1]:
The questions below concerns gravity in 5 spacetime dimensions and how it can be expressed as a theory in 4 spacetime dimensions. The starting point is the 5-dimensional equations (indicated by the hat-notation):

$$d\hat{s}^2 = \hat{g}_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}}, \tag{1.1}$$

and

$$\hat{g}_{\hat{\mu}\hat{\nu}} = e^{\phi/\sqrt{3}} \begin{pmatrix} g_{\mu\nu} + e^{-\sqrt{3}\phi} A_\mu A_\nu & e^{-\sqrt{3}\phi} A_\mu \\ e^{-\sqrt{3}\phi} A_\nu & e^{-\sqrt{3}\phi} \end{pmatrix}. \tag{1.2}$$

Here the hatted indices run over $\hat{\mu} = 0, 1, 2, 3, 4$ which we can express as the 4+1 split $\hat{\mu} = \mu, 4$. For the coordinates we write $\hat{x}^{\hat{\mu}} = (x^\mu, y)$ where we have renamed $\hat{x}^{\hat{4}}$ as y.
a) What is the main difference between the ideas of Kaluza and those of Klein?
b) Explain how one arrives at the relation $2\pi\kappa^2 = m\hat{\kappa}^2$ between the Newton constant in 4 dimensions (κ) and the one in 5 dimensions ($\hat{\kappa}$) (see eq. 8). What is m in this equation? Note: $8\pi G_N := \kappa^2$ in d=4 with a similar equation in d=5.
c) What are the changes needed in the ansatz for the metric in 5 dimensions (eq. 2) to lead to fields in 4 dimensions with canonical dimensions? You can use the fact that the metric is dimensionless in any dimension and that vector and scalar fields have dimension 1/L in natural units (see BZ p. 177 and p. 60-61)) in four dimensions. Recall that the action is always dimensionless (in natural units) and hence that the dimension of Newton's constant depends on the dimension of spacetime. Note that in eq. 8 in Duff's paper the κ^2 is multiplying the whole action which means that the vector and scalar fields in this equation do not have canonical dimensions (in accord with eq. 2).
d) The compactified theory contains charged fields and a $U(1)$ gauge theory (Maxwell's theory). What is the d=5 origin of the gauge symmetry?
e) How does the Kaluza-Klein theory explain the quantisation of electric charge?
Comment: This so called **Kaluza-Klein compactification** shows that the higher Fourier modes of the fields in 5 dimensions have non-zero masses (m_n) and charges (e_n) both given by the radius of the extra dimension. In fact, in Duff's paper he gives the relations

$$m_n = |n|m, \;\; e_n = n\sqrt{2}\kappa m, \;\;\; n \in \mathbf{Z}. \tag{1.3}$$

In most discussions about extra dimensions in this course we can think of them as having a size close to the Planck length $L_P = 1.6 \times 10^{-35}m$. One reason for this can be seen from the above expression for the charge (how?).

[1]If you are interested you may read Klein's original article in Nature 118 (1926) p. 516.

1.1 BZ Chapter 1: Introduction to string/M-theory

Let us start by defining the concept **unification**:
This refers to the process of collecting a number of different phenomena, formulas, etc and express them as consequences of some more general concepts thereby reducing the number of basic concepts and formulas. This is also referred to as "reductionism" discussed in detail by Steven Weinberg in his book "Dreams of a final theory".

Question: This seems like an obvious goal but what about the human mind, life etc? These deeper aspects of nature are discussed by Roger Penrose in his book "The emperor's new mind".

String theory is a natural next step in the following sequence of historical unifications:
1687: Newton's "Principia",
1865: Maxwell's equations,
1905: Einstein's special relativity,
1914: Einstein's theory of general relativity,
Late 1960's: The electroweak theory in the Glashow-Weinberg-Salam model (Nobel prize 1979).

Comment: Einstein also tried to unify gravity and electromagnetism by defining a new field

$$G_{\mu\nu} := g_{\mu\nu} + F_{\mu\nu}, \tag{1.4}$$

which does not work! Why is this not a good idea? In string theory this step is natural but then $F_{\mu\nu}$ is replaced by $B_{\mu\nu}$, the so called Kalb-Ramond field which has nothing to do with electromagnetism as we will see later.

Kaluza-Klein theory: A unification of gravity and electromagnetism can be accomplished if one introduces "higher dimensions", here ordinary general relativity in 5 dimensions as done by Kaluza in 1919 and Klein in the 1920s. The Swedish physicist Oskar Klein was very close to discovering non-abelian gauge theory (in fact SU(2)) which was not found until 1954 by Yang and Mills. Einstein was the referee on Kaluza's paper and he was very sceptical!

"Extra dimensions", i.e., dimensions not among the standard three space and one time, can not immediately be dismissed as nonsense. This becomes obvious by looking at the experimental status of such dimensions: see Hoyle et al, hep-th 0011014: Compact dimensions smaller than about a micron are still beyond experimental observation using gravity! Please have a look at this paper and try to verify this statement about the smallest possible extra dimension detectable. Note that the electron is known to be point-like down to about $10^{-20}m$. The reason why there can be two different smallest scales will be very important later when we discuss how the Standard Model can be extracted from string theory (Part 2 of BZ).

String theory predicts that physics takes place in ten dimensions (with Lorentzian signature) and **M-theory** that it is in eleven dimensions. M-theory is strongly believed to be a completely unique theory containing only one free parameter (with dimension length). String theory on the other hand comes in several versions but they can all be seen to be just various limits of M-theory. These fundamental aspects will be briefly elaborated upon later in the course. The fact that string/M-theory predicts the dimension of spacetime is why Kaluza-Klein theory and "compactifications" are so extremely important to study: We must of course at the end of the day derive physics as we see it in four dimensions which forces us to somehow "eliminate" the extra dimensions between four and ten or eleven.

Other possible unifications are:
1970s: A further, but not established, step in this direction is "GUT" theory (Grand Unified Theory). If true GUT unifies all the three non-gravitational forces in the Standard Model (SM) of elementary particles, i.e., the gauge theories based on the the groups $U(1) \times SU(2) \times SU(3)$ (i.e., EM plus the weak and the strong nuclear forces) into a single Yang-Mills theory based on $SU(5)$ or some even bigger gauge group. In string theory the group E_8 plays a special role in this context.

Quantum mechanics: Another fundamental result in this sequence of unifications involves QM:
1970s: 't Hooft (Nobel prize 1999) proved that Yang-Mills theory (YM) is renormalisable, i.e. that YM is consistent as a quantum field theory. This is not the case for Einstein's general relativity.

The main result of string/M-theory is that it predicts both the existence of the graviton (or rather "postdicts") as a consequence of quantum mechanics and explains how Einstein's general theory of relativity can be improved to give a theory without any infinite Feynman diagrams. Such a theory is called a **quantum gravity theory**. String theory also contains the Standard Model if the compactification (see Duff in Hp 1) to four dimensions is done properly (more later). In this sense string/M-theory therefore unifies particles with all spins from 0 to 2 including half-integer ones in a way compatible with quantum mechanics. At a deeper level it also seems to unify particles with objects like solitons[2], a feature related to what is called **duality**.

What is string theory: There is no known principle, like general coordinate invariance in GR, that gives rise to string theory. String theory is therefore usually obtained by constructing its perturbation theory which can be done in a unique way. Although we will not get that far in this course we will develop the theory to the point where we can start addressing this issue.

[2]If you want more information on this issue you can have a look at the review article by Joe Polchinski "Dualities of fields and strings" hep-th/1412.5704.

In more philosophical terms it seems that string/M-theory is actually not a theory (like GR and the Standard Model) but rather a **framework** similar to quantum mechanics. Thus we should perhaps instead think of string theory as a generalisation of quantum mechanics that can also be applied to gravity. Recall that Einstein's theory of gravity (i.e., GR) is not consistent with QFT which, however, string/M-theory is. How this works will be briefly explained later in the course.

Another nice property of string/M-theory is that it introduces new bridges to mathematics. In fact, some of the modern areas of mathematics started with ideas in string theory which then were developed both by physicists and mathematicians often in joined collaborations.

Question: Why was such a strange theory based on fundamental strings developed in the first place?
Answer: Historically, the string was discovered by accident from hadron physics and QCD. In a meson one can not pull the quark and the anti-quark apart to make them move freely. This is called *confinement* and means that if you increase the distance (by adding energy) between the quark and the anti-quark the gluon field starts behaving as a thin tube of field lines with properties similar to a rubber band, or an elastic piece of string with a quark at each end. If enough energy is added the QCD-string will snap and produce two more ends that will be associated with a new $q - \bar{q}$ pair resulting in two mesons. When the idea of a QCD-string was developed in the 70's one soon discovered that the spectrum contained a *massless spin-2 particle* which had to be identified with the graviton (see, e.g., the famous paper by Gliozzi, Olive and Scherk from 1976). This fact turned string theory into a theory of gravity and other fundamental fields.

The swampland program: There seems to be a huge number ($>> 10^{500}$) of possible compactifications turning the 10-dimensional string theory into a field theory in four spacetime dimensions. These four-dimensional field theories are then said to be part of the *landscape*. All other possible field theories that we can construct directly in four dimensions, like the Standard Model, might then not be compatible with quantum gravity (i.e., not derivable from string theory). If not they end up in the *swampland*. In recent years a lot of research has been done trying to formulate criteria that will tell us if a given four-dimensional field theory belongs to the swampland or not. The fact that our universe is in a de Sitter phase is a problematic issue that is addressed in the swampland program.

1.2 BZ Chapter 2: Special relativity and extra dimensions

Below we will discuss
1. Units and parameters in physics
2. Light-cone coordinates
3. Extra dimensions: Lorentz invariance, compact dimensions, orbifolds and how energy spectra can depend on compact dimensions.

Units and parameters

The following very basic facts are quite important in string theory:

1. There are only three basic units in nature: kg (mass), m (length) and s (time). These units will generally be denoted as M, L, T. Quantities (e.g., ds^2 in GR) then have dimensions given by powers of M, L, T written with a bracket as $[ds^2] = L^2$. The unit N (Newton) means that $[Force] = ML/T^2$.
All other units, e.g., the one related to charge, can be expressed in terms of these three basic units.

2. There are (exactly) two fundamental parameters relating them[3]:
c (m/s) : the velocity of light
$\hbar$ (Nms=kg m^2/s): Planck's constant
$\Rightarrow$ Hence only one of kg, m and s is independent.

Natural units: Defined by setting $c = \hbar = 1$ which means that $T = L$ and $M = 1/L$ (see BZ p. 177 - 178).

The choice of units is a messy issue. Three often used sets of units are
1. Gaussian units (uses CGS (centimeters, grams and seconds)) where the electrostatic force is given by

$$F = \frac{q_1 q_2}{r^2}. \tag{1.5}$$

Here electric charge has dimension, $[q] = esu$, with $(1\ esu)^2 = 10^{-5} N \cdot (10^{-2} m)^2 = 10^{-9} N m^2$. This implies that esu has unit $\frac{kg^{1/2} m^{3/2}}{s}$ which is not a new unit.
2. SI units: Here the electrostatic force reads

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}. \tag{1.6}$$

Charge has here dimension $Coulomb$ ($[q] = C$) where $1C = 1As$ with 1 ampere (A) defined as the current in two wires 1 m apart effecting each other by a force $F = 2 \cdot 10^{-7} N$ per

[3]These are today exactly defined numbers, see e.g. Wikipedia "2019 redefinition of the SI base units": 1 second = 9192631770 periods of the radiation of a specific transition in Ce^{133} while c is exactly 299792458 m/s and h is exactly $6.62607015 \cdot 10^{-34} kgm^2/s$. Thus c and $\hbar$ are numerically fixed and can not depend on either time or space. However, this last fact has been debated a bit in the past, see e.g. M.J. Duff, hep-th/0208093 and hep-th/0110060.

meter of the wires. Coulomb is just an auxiliary unit since it actually cancels in the force law since the constant ϵ_0 has dimension $[\epsilon_0] = C^2/Nm^2$.
3. We will in this course use the **Heaviside-Lorenz** system of units (BZ p. 45) with the definition of charge:

$$F = \frac{q_1 q_2}{4\pi r^2}. \tag{1.7}$$

Here the charge has the same unit as in 1) above, i.e., Nm^2, but its numerical value differs from esu due to the 4π factor in the force equation. Note that in three space dimensions esu is dimensionless in natural units $[esu] = L^0$ (more on this below)!

Intervals and Lorentz transformations

Here we will specify some notation and give some of the conventions used throughout the course. First, coordinates are denoted

$$x^\mu = (ct, x, y, z) := (x^0, x^1, x^2, x^3), \tag{1.8}$$

which are coordinates in spacetime, either flat or curved. The metric is "mostly plus", i.e., the Minkowski metric is $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$.

In BZ the interval is denoted ds and defined by

$$-\, ds^2 := -(ds)^2 = \eta_{\mu\nu} dx^\mu x^\nu = dx^\mu dx_\mu. \tag{1.9}$$

(This interval is sometimes denoted $d\tau$ as, e.g., in Weinberg's book on general relativity.) Note that we call

$$ds^2 > 0 \quad \text{time-like}, \tag{1.10}$$
$$ds^2 = 0 \quad \text{light-like}, \tag{1.11}$$
$$ds^2 < 0 \quad \text{space-like}. \tag{1.12}$$

Lorentz transformations are as usual given by (often denoted $\Lambda^\mu{}_\nu$)

$$x'^\mu = L^\mu{}_\nu \, x^\nu, \tag{1.13}$$

where, if the primed inertial frame moves with velocity v in the x-direction of the unprimed system

$$L^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{where } \beta = v/c \text{ and } \gamma = 1/\sqrt{1-(v/c)^2}. \tag{1.14}$$

These are the transformations that leave the interval invariant, i.e., $(ds')^2 = (ds)^2$. This fact is equivalent to the invariance of the metric, i.e.,

$$\eta_{\rho\sigma} L^\rho{}_\mu L^\sigma{}_\nu = \eta_{\mu\nu} \Leftrightarrow \; L^T \eta L = \eta, \tag{1.15}$$

where in the last equation we use matrix notation. Since these transformations leave a scalar product in a 4-dimensional Minkowski spacetime invariant the group they belong to is called $SO(1,3)$. We can turn this around and define Lorentz transformations as those coordinate transformations that leave the Minkowski metric numerically invariant. This discussion will in this course often be applied to spacetimes with more dimensions than four.

Comment: For light we have that $ds^2 = 0$. If we now ask what transformations leave $ds^2 = 0$ invariant one finds that the Poincaré group leaving $ds^2 \neq 0$ invariant is extended by some new transformations, one being *dilatations* (that is, scale transformations) $x'^\mu = \lambda x$ where λ is a spacetime independent parameter. Analysing this in detail leads to the conclusion that there appears a new group consisting of the following generators: Lorentz transformation $J^\mu{}_\nu$, a non-trivial set of spacetime translations also denoted P^μ, and the new dilatation generator D and special conformal generators K^μ. The group generated by these 10 generators turns out to be $SO(2,4)$, which is the *conformal group* in four-dimensional Minkowski space. However, it is also the isometry group of AdS_4 as we know from the gravity course. This extremely important fact is, as we will see in Chapter 23, the starting point for understanding the so called **AdS/CFT correspondence** (found by J. Maldacena in 1997).

Light-cone coordinates

This is perhaps the first concept that may be unfamiliar to you. To define light-cone coordinates we first give the definition and then discuss why they are useful. We will divide the set of coordinates x^μ into two sets x^0, x^1 and the rest x^2, x^3 (which in a general spacetime dimension $D = d + 1$ becomes $x^2, x^3, ..., x^d$). This last subset of the coordinates are called *transverse*. Next we define the light-cone coordinates (x^+, x^-, x^2, x^3) by combining x^0 and x^1 into

$$x^+ := \frac{1}{\sqrt{2}}(x^0 + x^1), \;\; x^- := \frac{1}{\sqrt{2}}(x^0 - x^1). \tag{1.16}$$

This definition implies a number of things:
1. The invariant interval becomes (just use that $2x^+x^- = (x^0)^2 - (x^1)^2$)

$$-ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -2dx^+dx^- + (dx^2)^2 + (dx^3)^2. \tag{1.17}$$

This can be written

$$-ds^2 = \hat{\eta}_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu, \tag{1.18}$$

where

$$\hat{x}^\mu := (x^+, x^-, x^2, x^3), \tag{1.19}$$

and

$$\hat{\eta}_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{which means} \quad \eta_{++} = \eta_{--} = 0 \;\text{ and }\; \eta_{+-} = \eta_{-+} = -1. \tag{1.20}$$

Question: Is the transformation between x^μ and $\hat{x}^\mu$ a Lorentz transformation?

2. In these light-cone coordinates there is no clear distinction between x^+ and x^- so which one is "time"? This is a matter of definition and we will choose x^+ as the light-cone "time". Thus x^- is a light-cone "space" coordinate. Note the unfamiliar-looking relation $x_- = \eta_{-+}x^+ = -x^+$. To avoid sign mistakes we will always use upstairs $+$ and $-$ indices, that is x^+ and x^-.

3. If we consider a particle with constant velocity v in the x^1-direction we will discover some peculiar properties of light-cone coordinates:
Consider a particle trajectory given by ($\beta := v/c$)

$$x^0(t) = ct = x^0, \;\; x^1(t) = vt = \beta x^0, \;\; x^2(t) = x^3(t) = 0. \tag{1.21}$$

These equations imply

$$dx^+ = \frac{1}{\sqrt{2}}(dx^0 + dx^1) = \frac{1}{\sqrt{2}}dx^0(1+\beta), \tag{1.22}$$

$$dx^- = \frac{1}{\sqrt{2}}(dx^0 - dx^1) = \frac{1}{\sqrt{2}}dx^0(1-\beta), \tag{1.23}$$

and hence the light-cone velocity becomes

$$\hat{\mathbf{v}} = (\hat{v}^-, \hat{v}^2, \hat{v}^3) = (\frac{dx^-}{dx^+}, \frac{dx^2}{dx^+}, \frac{dx^3}{dx^+},) = (\frac{1-\beta}{1+\beta}, 0, 0). \tag{1.24}$$

This result is a bit strange since $-1 < \beta < 1$ and thus $\infty > \hat{v}^- > 0$, so it has no limit (like in non-relativistic physics) but what is perhaps even more weird; it is never negative. These strange features one just has to accept.

4. Recall the standard relations in special relativity:

$$\text{4-velocity:}\, U^\mu = c\frac{dx^\mu(s)}{ds} \Rightarrow \text{4-momentum}\, P^\mu = mU^\mu. \tag{1.25}$$

We also have in special relativity the interval

$$ds = \sqrt{ds^2} = \sqrt{c^2dt^2 - (d\mathbf{r})^2} = cdt\sqrt{1-\beta^2}, \tag{1.26}$$

which implies that, using $P^\mu = (E/c, P^1, P^2, P^3)$,

$$P^\mu = \frac{m}{\sqrt{1-\beta^2}}(c, \mathbf{v}) \;\Rightarrow\; E = \gamma mc^2, \;\; \mathbf{P} = \gamma m\mathbf{v}. \tag{1.27}$$

We also see that

$$P^2 = P^\mu P_\mu = -m^2c^2. \tag{1.28}$$

Having reviewed the standard relations we can now answer the following question:
What is the generator of translations in the light-cone "time" coordinate x^+?

In QM we know that the unitary translation operator is

$$U(t, \mathbf{r}) = e^{\frac{i}{\hbar} x \cdot \hat{p}} \tag{1.29}$$

In ordinary coordinates the exponent reads

$$x \cdot p = -x^0 p^0 + x^1 p^1 + x^2 p^2 + x^3 p^3 = -tE + \mathbf{r} \cdot \mathbf{p}. \tag{1.30}$$

We know (for instance, from the Schrödinger equation) that the operator corresponding to the energy E is the Hamiltonian so $\hat{H} = c\hat{p}^0$ is the generator of time translations.

In light-cone coordinates the corresponding equation (dropping the hats) is

$$x \cdot p = x^+ p_+ + x^- p_- + x^2 p_2 + x^3 p_3 = -x^+ p^- - x^- p^+ + x^2 p_2 + x^3 p_3, \tag{1.31}$$

which means that H is replaced by

$$H_{l.c.} = cp^-. \tag{1.32}$$

This fact then gives the light-cone version of the Schrödinger equation. Later we will have problems dealing with the square root solution of $p^2 = -m^2$, $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$, which in light-cone coordinates is replaced by the much simpler $p^- = \frac{1}{2p^+}((\mathbf{p}^T)^2 + m^2)$. Here T refers to the transversal directions $p^\mu \neq p^\pm$ and we assume $p^+ \neq 0$.

Extra dimensions and their effect on energy spectra

First two comments:

1. More time directions than one is not easy to make sense of in physics (but there have been many attempts).
2. Having more space directions than three is (fairly) easy to deal with and they are actually predicted by the string. Moreover, as seen from the paper by Hoyle et al (hep-th/0011014), extra space dimensions with size around a micron or smaller can not yet be ruled out by gravitational experiments. The possible existence of micron sized extra dimensions is quite remarkable.

What would a spacetime with extra space dimensions mean?

As an example consider a 6-dimensional spacetime with coordinates $x^M = (x^0, x^1, x^2, x^3, x^4, x^5) = (x^\mu, y^m)$. Here we have introduced the notation $y^m = (y^1, y^2) = (x^4, x^5)$ for the dimensions added to our ordinary 4-dimensional spacetime. Thus we have the 4+2 index split $M = (\mu, m)$. (If needed one could have used a hat: $\hat{x}^M$.)

Almost all properties of ordinary spacetime are easily generalised to higher dimensions:

1. SO(1,3) $\rightarrow$ SO(1,5) which means that the invariant interval is now

$$-ds^2 = -c^2(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 + (dx^5)^2 = \eta_{MN} dx^M dx^N, \tag{1.33}$$

where we also used the higher dimensional Lorentz metric

$$\eta_{MN} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.34}$$

Here we see that the upper left hand 4×4 block is just $\eta_{\mu\nu}$ and that in the extra directions the metric is $\eta_{mn} = \delta_{mn}$. All mixed components are zero ($\eta_{\mu n} = 0$).

In a similar spirit we can decompose the Lorentz rotation matrix:

$$L^M{}_N = \begin{pmatrix} L^\mu{}_\nu & L^\mu{}_n \\ L^m{}_\nu & L^m{}_n \end{pmatrix} . \tag{1.35}$$

The interpretation of these four blocks is clear:
$L^\mu{}_\nu$ is just the ordinary Lorentz rotations in four dimensions SO(1,3),
$L^m{}_n$ are rotations in two Euclidean dimensions i.e. SO(2),
$L^\mu{}_n$ and $L^m{}_\nu$ mix the 4-dim spacetime directions and the extra ones.

Properties of extra dimensions:
1. In field theory we consider only extra dimensions that are smooth manifolds (having no singular points) so only torii, spheres and some other more complicated (but smooth) mathematical manifolds are possible. In five dimensions the 5th direction can then only be a circle. In this case the whole spacetime can be viewed as a generalised "cylinder": The uncompactified dimension along the cylinder then has a circle attached to it at each point.

There are two different ways to describe a circle:

a) As a circle with an angle coordinate $0 \le \theta < 2\pi$. This coordinate is single-valued over the circle but it is discontinuous: It jumps from 2π back to 0 after one revolution.

b) Instead one can use a multi-valued but continuous coordinate y if one unwraps the circle so that it covers the whole y-axis from $-\infty$ to $+\infty$. In this case the circle is the **fundamental region** under the equivalence relation

$$\text{circle}: \; y \sim y + 2\pi R, \tag{1.36}$$

where R is the radius of the circle.

In string theory it is often easier to use the second kind of coordinates, i.e., multivalued and continuous ones!

Example in 2 dimensions: By dividing the plane with coordinates (x, y) into squares and identifying all opposite sides one ends up with a fundamental region which is a flat two-dimensional torus. Such a torus can not be embedded in $\mathbf{R}^3$.

2. In string theory the above manifolds can be generalised to **orbifolds** which is not possible in field theory. Orbifolds are singular and thus not really manifolds. Such objects arise as the fundamental region due to a transformation with fix-points. The simplest example is the cone: Consider the plane again with coordinates (x, y). The torus above was obtained by using equivalence relations based on translations in the two directions (as for the circle) which have no fixpoints: Translations are effective everywhere on the plane.

Use instead polar coordinates (r, θ) and consider the following equivalence relation:

$$\theta \sim \theta + \pi/2. \tag{1.37}$$

Identifying the two sides of the segment between $\theta = 0$ and $\theta = \pi/2$ gives a cone with a singular point at the tip. This tip arises because the equivalence relation has no effect at the origin of the plane.

Energy levels and compact dimensions: As a final point in this chapter we discuss how the energy spectrum of a physical system depends on the presence of an extra circular dimension. This will explain the importance of understanding the different energy scales that occur in theories with extra dimensions and how it might be possible to detect such compact extra dimensions. Experiments of this kind are currently conducted at CERN (see ATLAS, June 2016[4]) and many other large colliders.

To understand this phenomenon in a very simple setting let us consider the Schrödinger equation for a particle in a square well potential in one dimension (x) studied in every basic course on quantum mechanics. To get the energy levels we need to solve

$$\left(\frac{p^2}{2m} + V(x)\right)\psi(x) = E\psi(x), \;\; \text{where} \;\; p = -i\hbar\frac{d}{dx}. \tag{1.38}$$

The potential is given by

$$V = 0, \;\; 0 < x < a,$$

$$V = \infty \;\; \text{for} \;\; x \leq 0 \;\; \text{and} \;\; x \geq a \;\Rightarrow\; \psi(x) = 0 \text{ in this range.}$$

The equation to solve inside the square well is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi, \tag{1.39}$$

giving, with $\int_0^a |\psi_k(x)|^2\, dx = 1$, for each integer $k = 1, 2, ...,$

$$\psi_k(x) = \sqrt{\frac{2}{a}}\sin(\frac{k\pi x}{a}), \quad E_k = \frac{\hbar^2}{2m}\left(\frac{k\pi}{a}\right)^2. \tag{1.40}$$

[4] More details can be found concerning searches for both supersymmetry and extra dimensions in the 2017 PhD thesis by G. Bertoli from Stockholm University.

We now solve the same problem but in a space with an extra circular direction with coordinate y. So $x \in \mathbf{R}$ and $y \in S^1_R$ where R is the radius of the circular dimension. Outside the well in the x-direction the wave function is zero for all values of y. Inside the well we must solve the equation

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi(x,y)}{\partial x^2}+\frac{\partial^2\Psi(x,y)}{\partial y^2}\right)=E\Psi(x,y). \tag{1.41}$$

Here the variables x and y can be separated by writing the wave function as

$$\Psi(x,y)=\psi(x)\phi(y), \tag{1.42}$$

which gives the answer that $\psi(x)$ is the same as before (the $\psi_k(x)$ quoted above) while

$$\phi_l(y)=a_l\sin(\tfrac{ly}{R})+b_l\cos(\tfrac{ly}{R}). \tag{1.43}$$

Note that the expansion on the circle contains both sin and cos modes while on the interval $[0,a]$ in the x-direction only sin modes can appear since the wave function has to vanish at $x=0$ and $x=a$. Solving the Schrödinger on the cylinder space gives

$$E_{k,l}=\frac{\hbar^2}{2m}\left((\frac{k\pi}{a})^2+(\frac{l}{R})^2\right),\;\; k=1,2,3,...\;\;,l=0,1,2,3,... \tag{1.44}$$

Note the absence of the $k=0$ mode!

This result is very interesting when trying to understand the relevance and detectability of extra dimensions. In a physical situation we can let a be the size of an atom, $10^{-9}m$, and R the Planck length, about $10^{-35}m$. So $a>>R$ and hence $\frac{\pi}{a}<<\frac{1}{R}$. The conclusion is that it is easy to see the excited k-levels but the first excited l-level requires an energy close to the Planck energy $10^{19}GeV$ to see. This is an energy that never will be produced in any collider on Earth. It is therefore a big challenge to design experiments that can circumvent this problem. The modern approach to confront string theory with experiments is via the **swampland program**. More on this at the end of the course if time permits.

Comment: As we will see later, a closed string living on the cylinder, compared to the particle above, will have more degrees of freedom corresponding to the fact that it can **wind** any number of times around a non-trivial direction like the circle in this case. A space with no non-trivial loops is called **simply connected** and one with non-trivial loops **non-simply connected**. This new feature of strings living on such non-trivial spaces is very important in string theory and we will return to it later in this course.

Field theory on a "cylinder": What happens to a field theory in $Mink_4$ if we add a circular dimension S^1 at each point? As a simple example consider a free Klein-Gordon scalar field in five spacetime dimensions with one compact dimension. Replacing $Mink_5$ by $Mink_4\times S^1$ with coordinates (x^μ,y) then means that

$$\Box_5\phi(x^\mu,y)=0\Rightarrow\Box_4\phi(x^\mu,y)+\partial_y^2\phi(x^\mu,y)=0. \tag{1.45}$$

The mode expansion on a circle with radius R reads $(0 \leq y < 2\pi R)$

$$\phi(x^\mu, y) = \Sigma_{n\in\mathbf{Z}}\, \phi_n(x^\mu) e^{iny/R}. \tag{1.46}$$

It is clear that the five dimensional Klein-Gordon field gives rise to an infinite set of scalar fields $\phi_n(x^\mu)$ in four dimensions. The five-dimensional Klein-Gordon equation is now equivalent to the set of equations

$$\Box_4 \phi_n(x^\mu) - (\frac{n}{R})^2 \phi_n(x^\mu) = 0 \Rightarrow m_n = |n|\frac{1}{R}. \tag{1.47}$$

Once again we see the consequences of a compact dimension of Planck size: The fields with a Planck size mass are basically impossible to create in the lab. This means that in most approaches to deriving Standard Model type field theories from strings only the "massless" sector is of interest.

2 Lecture 2

2.1 BZ Chapter 3: EM and GR in various dimensions

Maxwell's equations in Heaviside-Lorentz units, in four dimensions with coordinates $(t, \mathbf{r}) = (t, x, y, z)$, read:

$$\nabla \cdot \mathbf{E} = \rho, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{2.2}$$

In these units the two fields $\mathbf{E}$ and $\mathbf{B}$ have the same dimension. The first equations are the dynamical ones while the last two are Bianchi identities. The sources are ρ, the charge density with dimension $[\rho] = \frac{esu}{L^3} = esu/volume$, and $\mathbf{j}$, the charge current density with dimension $[\mathbf{j}] = \frac{esu}{L^2 T} = esu \times v/volume$.

Note that $[\mathbf{E}] = [\mathbf{B}] = \frac{esu}{L^2}$ which follows from Gauss' law: $q = \int_{S^2} \mathbf{E} \cdot d\mathbf{a}$. Thus since esu is dimensionless in natural units these fields have dimension $1/L^2$ and the potentials $(\phi, \mathbf{A})$ have dimension $1/L$.

The Lorentz force law is $(\beta = v/c)$

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \beta \times \mathbf{B}), \tag{2.3}$$

Now we can introduce the potentials $(\phi, \mathbf{A})$ by solving the two Bianchi identities:

$$\nabla \cdot \mathbf{B} = 0 \;\Leftrightarrow\; \mathbf{B} = \nabla \times \mathbf{A}, \tag{2.4}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \;\Leftrightarrow\; \mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{2.5}$$

The potentials $\phi(t, \mathbf{r}), \mathbf{A}(t, \mathbf{r})$ are not uniquely specified by their relations to the field strengths $\mathbf{E}, \mathbf{B}$. In fact, if we perform the **gauge transformations**

$$\phi'(t, \mathbf{r}) = \phi(t, \mathbf{r}) - \tfrac{1}{c}\dot{\epsilon}(t, \mathbf{r}), \;\; \mathbf{A}'(t, \mathbf{r}) = \mathbf{A}(t, \mathbf{r}) + \nabla\epsilon(t, \mathbf{r}), \tag{2.6}$$

where $\dot{\phi} = \partial_t \phi$, the parameters $\epsilon(t, \mathbf{r})$ cancel when computing $\mathbf{E}$ and $\mathbf{B}$.

Exercise: Check this last statemant about the gauge invariance.

Another way to express the physical content of gauge invariance is to say that the two sets of potentials $(\phi(t, \mathbf{r}), \mathbf{A}(t, \mathbf{r}))$ and $(\phi'(t, \mathbf{r}), \mathbf{A}'(t, \mathbf{r}))$ are physically equivalent! Note that there are situations where the potentials must be used, e.g., in the Aharonov-Bohm effect in QM and the covariant derivative in QFT.

Comment: Potentials defined on non-trivial spaces pick up some new very interesting properties. Such spaces are, e.g., non-simply connected ones like spaces with compact

circle dimensions or spaces with a wire with a current. Note that the magnetic field at the position of the wire is infinite and therefore not part of the space anymore (recall the Aharonov-Bohm effect). The following may happen:
1. On a circle different vector potentials are not always physically equivalent since any constant vector potential around a non-trivial circle gives rise to a gauge invariant **Wilson loop** $= e^{\oint \mathbf{A}\cdot d\mathbf{r}}$ (more on this later).
2. While the physically measurable field strengths $\mathbf{E}, \mathbf{B}$ must be uniquely defined everywhere on any spacetime (also non-trivial ones) this is not the case for the potentials $(\phi(t, \mathbf{r}), \mathbf{A}(t, \mathbf{r}))$. From QM we may recall that $\mathbf{A}(t, \mathbf{r})$ for a magnetic monopole is not unique over the whole space $\mathbf{R}^3$ around the monopole. In fact, on a 2-sphere around the monopole one has to give $\mathbf{A}(t, \mathbf{r})$ different values typically on the northern and southern hemi-spheres related by gauge transformations.

EM in general dimensions: Our main objective now is to find out what happens to electromagnetism given by Maxwell's equations if we change the number of *space* dimensions, either to only two (or even one) or to more than three. We will later need theories similar to the Maxwell theory in one time and nine or even ten space dimensions if gravity is involved (discussed briefly if time permits).

The case of two space dimensions is discussed in BZ but here we turn directly to the general case. The reason for this is that if we formulate the equations above relativistically then they are automatically correct in any spacetime dimension with Lorentz symmetry SO(1,d) where d is the number of space dimensions. Thus we rewrite the above equations as follows (which should be well-known from any course in special relativity):

$$A^\mu = (\phi, A^i), \;\; A_\mu = (-\phi, A^i), \tag{2.7}$$

where it is important to note that the space components (the vector potential $\mathbf{A}$) are here written with a space index $i = 1, 2, ..., d$, i.e., as A^i. Thus we use the index split $\mu = (0, i)$ in any number of dimensions. Then derivatives in four dimensions (∂_0, ∇) become

$$\partial_\mu := \frac{\partial}{\partial x^\mu} = (\partial_0, \partial_i), \tag{2.8}$$

and the field strength reads

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.9}$$

The four-dimensional $\mathbf{E}, \mathbf{B}$ defined by $F_{0i} = -E_i$ and $F_{ij} = \epsilon_{ijk} B^k$ then generalise in higher (or lower) dimensions to

$$F_{0i} = -E_i, \;\; F_{i_1 i_2} = \epsilon_{i_1 i_2 i_3 ... . i_d} B^{i_3 ... i_d}. \tag{2.10}$$

We see directly that the number of components of the electric field strength follows the number of space dimensions but that the number of magnetic components increases much more rapidly: in 9 space dimensions we have 9 electric but $[ij]$ magnetic ones, that is $9 \cdot 8/2 = 36$. In two space dimensions this gives two electric components but only one magnetic one. In higher dimensions it is better to use $F_{i_1 i_2}$ than $B^{i_3 \cdots i_d}$.

The other relativistic equations we need are the dynamical ones

$$\partial_\nu F^{\mu\nu} = \tfrac{1}{c} J^\mu, \text{ where } J^\mu = (c\rho, j^i) \tag{2.11}$$

and the Bianchi identities

$$\partial_{[\mu} F_{\nu\rho]} = 0, \tag{2.12}$$

all invariant under the gauge transformations

$$A'_\mu = A_\mu + \partial_\mu \epsilon, \tag{2.13}$$

and valid in a spacetime of any dimensionality.

When we now start to discuss the fields generated by point charges in a general spacetime (with d space dimensions) we will use Gauss' law and thus integrals over higher-dimensional spheres S^{d-1}. Note the relation between a **ball** B^d and a its boundary, the sphere S^{d-1}, is written

$$S^{d-1} = \partial B^d, \text{ where } \partial \text{ is the boundary operator satisfying } \partial^2 = 0. \tag{2.14}$$

The method used to obtain the volume of these spheres is standard (see BZ sect. 3.4 or the QFT course). The answer is

$$Vol(S^{d-1}_{unit}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{2.15}$$

The Γ-function is defined by $\Gamma(n+1) = n\Gamma(n)$ for integers n or as an integral for any real $x \in \mathbf{R}$

$$\Gamma(x+1) = \int_0^\infty dt e^{-t} t^x. \tag{2.16}$$

We can now check how the component equations generalise to higher dimensions. So starting from the fact that the dynamical equations $\partial_\nu F^{\mu\nu} = \frac{1}{c} J^\mu$ are true in any spacetime dimension $D = 1 + d$ we set $\mu = 0$ and find that Gauss' law

$$\partial_i F^{0i} = \frac{1}{c} J^0 \Rightarrow \nabla \cdot \mathbf{E} = \rho, \tag{2.17}$$

is valid in any dimension (here we have just used $F^{0i} = E^i$ and $J^0 = c\rho$). Then

$$[\rho] = esu/L^d \Rightarrow [\mathbf{E}] = esu/L^{d-1}. \tag{2.18}$$

But using the force equation $\mathbf{F} = q\mathbf{E}$ we also find that

$$[\mathbf{F}] = (esu)^2/L^{d-1} \Rightarrow esu = \sqrt{\frac{ML^d}{T^2}} = \sqrt{L^{d-3}}, \tag{2.19}$$

where the last equality follows if we use natural units. The conclusion is therefore that *charge is a dimensionless quantity only in 4 spacetime dimensions.*

Let us now repeat some standard manipulations in 3 space dimensions but now in d space dimensions. Integrating Gauss' law over a d-dimensional ball B^d gives the charge q inside the ball

$$q = \int_{B^d} \rho \, d(Vol) = \int_{B^d} \nabla \cdot \mathbf{E} \, d(Vol). \tag{2.20}$$

Applying Gauss' theorem (valid in any dimension, see BZ sect. 3.5) to the last expression as usual, it becomes (if the ball is round, i.e., $\partial B^d = S_r^{d-1}$),

$$q = \int_{S_R^{d-1}} \mathbf{E} \cdot d\mathbf{A} = \text{flux through the surface}\, S_r^{d-1}. \tag{2.21}$$

Here $d\mathbf{A}$ is the $(d-1)$-dimensional area element converted into a $(d-1)$-dimensional vector perpendicular to the surface in question. This is done using the ϵ-tensor in d-dimensional space: $(dA)^{i_1 \ldots i_{d-1}} \epsilon_{i_1 \ldots i_d} = dA_{i_d}$.

As usual we can now apply this to a point charge at the origin of d-dimensional space. To do this we use the form of the volume element expressed in terms of spherical coordinates $(r, \theta_1, ..., \theta_{d-2}, \phi)$ in d dimensions:

$$d(Vol_d) = dr \, r^{d-1} d\Omega_{d-1}. \tag{2.22}$$

This gives the electric field from a point charge $\mathbf{E} = (E_r(r), 0, ..., 0)$ as follows

$$q = \int_{S_r^{d-1}} \mathbf{E} \cdot d\mathbf{A} = E_r(r) \, r^{d-1} Vol(S_{unit}^{d-1}) = E_r(r) \, r^{d-1} \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{2.23}$$

So we finally find that

$$E_r(r) = \frac{\Gamma(d/2)}{2\pi^{d/2}} \frac{q}{r^{d-1}}. \tag{2.24}$$

This clearly generalises the usual Heaviside-Lorentz formula in three dimensional space: $d = 3$ implies with $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$ that $E_r(r) = \frac{q}{4\pi r^2}$.

Comment: A slightly different way to say this is to recall that in $d = 3$ space dimensions functions $f(r) = r^{-1}$ are called harmonic since they satisfy $\nabla^2 f(r) = 0$ for $r > 0$ and gives a delta function at the origin. This follows from the integrals above and that in spherical coordinates $\nabla^2 f(r) = r^{-2}\partial_r(r^2\partial_r)f(r)$. In d dimensions $\nabla^2 f(r) = r^{-(d-1)}\partial_r(r^{d-1}\partial_r f(r))$ so harmonic functions now behave as $f(r) \sim r^{-(d-2)} = r^{-d+2}$ which is what we saw above (remember that $\mathbf{E} = -\nabla\phi(r)$ where $\phi(r)$ is harmonic). Note that these expressions are obtained using $\sqrt{\det g_{ij}} \propto r^{d-1}$ as seen from the volume formula above.

Gravity in general dimensions: Our next task is to repeat the above considerations in the case of gravity. As we will see this is a lot more interesting than electromagnetism. In particular, the results will be needed later when we study the relation between gravity and string theory which is one of the key things to understand in this course. The issue is connected to the following question:

Question: As will become clear later string theory has only **one** free parameter, the **string length**, denoted l_s and having dimension L. String theory has no other free parameters, dimensionless or with dimension. This is quite remarkable since we eventually must be able to derive standard model type physics in 4 spacetime dimensions which today requires a QFT with about 20 free parameters. The parameter with dimension in field theory (in $D = 4$) is Newton's constant G (often denoted G_N) with dimension L^2 (recall the Lagrangian in GR) in natural units. So what is the relation between l_s and G? The answer depends on the dimension of spacetime so this is what we need to discuss first.

Consider the force law in Newtonian mechanics, now in a general dimension d. First we note that force, coordinates and masses have dimensions (N, m, kg) which are the same in all space dimensions. This means that the force in any dimension reads[5]

$$F = G^{(D)} \frac{m_1 m_2}{r^{d-1}} \times \frac{2\Gamma((D-1)/2)}{\pi^{(D-1)/2-1}}. \tag{2.25}$$

Here $G^{(D)}$ is the Newton constant in spacetime-dimension $D = d + 1$, so $G^{(4)} := G$. It is clear that Newton's constant has a dimensionality that depends on the dimension of spacetime due to the r^{d-1}-dependence of the force law (recall the result in electromagnetism). Thus

$$[G^{(D)}] = \frac{Nm^{D-2}}{kg^2} = \frac{L^{D-1}}{MT^2} = L^{D-2}, \tag{2.26}$$

where the last expression is in natural units.

We can now make the differences between electromagnetism and gravity clear by comparing the Newtonian force laws:

$$\mathbf{F}_e = q\mathbf{E}, \quad \mathbf{F}_g = m\mathbf{g}. \tag{2.27}$$

The force is independent of the spacetime dimension D, which is the case also for the mass m and thus also for the gravity field $\mathbf{g}$. In the electric case, on the other hand, both the charge q and the field $\mathbf{E}$ depend of D as we saw above.

It is very useful to define Planck size quantities in $D = 4$ as follows (here $[G] = L^2$):

$$\text{Planck length}: \;\; l_P = \sqrt{\frac{G\hbar}{c^3}} = 1.6 \times 10^{-35} m, \tag{2.28}$$

$$\text{Planck time}: \;\; t_P = \frac{l_p}{c} = \sqrt{\frac{G\hbar}{c^5}} = 5.4 \times 10^{-44} s, \tag{2.29}$$

$$\text{Planck mass}: \;\; m_P = \sqrt{\frac{\hbar c}{G}} = 2.2 \times 10^{-8} kg, \tag{2.30}$$

$$\text{Planck energy}: \;\; E_P = m_P c^2 = \sqrt{\frac{\hbar c^5}{G}} = 1.2 \times 10^{19} GeV. \tag{2.31}$$

[5]The last factor is designed to give 1 in $D = 4$ and thus $\nabla^2 V^{(D)} = 4\pi\rho^{(D)}$ in all dimensions D. The Einstein-Hilbert action is then also the same for all D.

Recall that $1\,eV := 1.6 \times 10^{-19} Nm$. Note that these scales are associated with quantum gravity. That is, at the energy $1.2 \times 10^{19} GeV$, or length scale $1.6 \times 10^{-35} m$, gravity must be treated as a quantum theory. Since Einstein's theory of gravity (GR) is not renormalisable (i.e., not consistent as a QFT) it has to be modified. String theory will explain exactly how this can be done!

Comment An important question in the formulation of the laws of nature is why the Planck scale is so much different from other scales, e.g., the scale of the weak nuclear force around 100 GeV, and the scale of neutrino physics around 0.1 - 1 eV. It is an amazing fact that the latter is very close to the scale of the curvature of the universe as given by the measured cosmological constant. The existence of several different scales in nature is called the **hierarchy problem** to which we have no really good answer.

It is convenient to define the Planck length in a general spacetime dimension:

$$l_P^{(D)} = \left(\frac{G^{(D)} \hbar}{c^3} \right)^{1/D-2}. \tag{2.32}$$

This may be expressed as

$$G^{(D)} = \frac{(l_P^{(D)})^{D-2}}{(l_P)^2} G. \tag{2.33}$$

This follows directly since $\frac{\hbar}{c^3} = \frac{l_P^2}{G}$ where G and l_P are the 4-dimensional quantities.

Compactified dimensions: Having understood the properties of the gravitational constant and the Planck length in general dimensions we should now try to relate them to each other when some directions of spacetime are compactified, via some Kaluza-Klein procedure. The simplest case to look at is to let the compact dimensions be circles, that is, a **torus-compactification**.

The simplest possible torus-compactification is obtained if the radius of all the circles are the same, say R. We then consider the Lagrangian for gravity in D dimensions written on a spacetime with 4 ordinary (non-compact) dimensions and n circular ones (an n-torus): thus $D = 4+n$. Then the Einstein-Hilbert action can be written (with an explicit Newton's constant $G^{(D)}$ and $\mathcal{L} := \sqrt{-g}R$)

$$S^{(D=4+n)} = \frac{1}{16\pi G^{(D)}} \int d^4x \left(\int_0^{2\pi R} dy_1 ... \int_0^{2\pi R} dy_n \right) \mathcal{L}(g_{MN}(x,y)). \tag{2.34}$$

Expanding the D-dimensional metric in Fourier-components on n circles is not hard to do but when inserted into $S^{(D=4+n)}$ it gives rise to a rather complicated expression. However, here we are only interested in deriving the relation between $G^{(D)}$ and G so the only term we need to look at in the final result in 4 dimensions is the Einstein-Hilbert one. Hence, we first do a $4+n$ split $M = (\mu, m)$ and set to zero all higher Fourier modes of the

4-dimensional fields:

$$g_{MN}(x,y) \to g_{\mu\nu}(x,y), A_{\mu n}(x,y), \phi_{mn}(x,y) \to g_{\mu\nu}(x), A_{\mu n}(x), \phi_{mn}(x). \tag{2.35}$$

In a second step we also drop the zero modes $A_{\mu n}(x), \phi_{mn}(x)$ leaving only $g_{\mu\nu}(x)$. Then we can do all the y-integrals trivially

$$S^{(D=4+n)} \to \frac{(2\pi R)^n}{16\pi G^{(D)}} \int d^4x \mathcal{L}(g_{\mu\nu}(x)) := \frac{1}{16\pi G} \int d^4x \mathcal{L}(g_{\mu\nu}(x)). \tag{2.36}$$

The definition of the $D = 4$ Newton's constant in the last equality leads to the relation

$$G^{(4+n)} = (2\pi R)^n \, G. \tag{2.37}$$

This is sometimes expressed in terms of the compactification length given by $l_c := 2\pi R$. Then for a torus compactification we have

$$G^{(D)} = (l_c)^{D-4} G. \tag{2.38}$$

This result can also be obtained by considering, as in BZ, the Poisson equation in some higher dimensional space with n torus dimensions where the mass source is spread evenly over the compact dimensions. The mass density then becomes

$$\rho^{(4+n)} = \frac{\rho}{(2\pi R)^n}. \tag{2.39}$$

The above G-result then follows from the fact that for the Poisson equation in this situation the right hand side is dimension independent, i.e., $G^{(D)}\rho^{(D)} = G\rho$. (Verify this statement!)

Having obtained the two equations $G^{(D)} = \frac{(l_P^{(D)})^{D-2}}{(l_P)^2} G$ and $G^{(D)} = l_c^{(D-4)} G$ above we can now play around with dimensions and check some scale relations. An interesting question is how big the compact dimensions can be. To address this question we eliminate both Newton constants from the two equations above. We get, for an internal torus with all directions of the same size l_c,

$$(l_P^{(D)})^{D-2} = (l_P)^2 (l_c)^{D-4}. \tag{2.40}$$

Let's consider a couple of examples. For $D = 5$ this relation reads

$$(l_P^{(5)})^3 = (l_P)^2 \, l_c. \tag{2.41}$$

If the smallest length structure we can detect today, namely about $10^{-20}m$ (using 10 TeV beams at CERN)[6], is assumed to be the one relevant for the Planck length in $D = 5$ then we find, using $l_P = 10^{-35}m$, that

$$l_c = \frac{10^{-60}m}{10^{-70}m} = 10^{10}m, \tag{2.42}$$

[6] The relation between length scales and energies scales is most easily obtained from the observation that $l_p E_P = \hbar c \approx 10^{-16} m \cdot GeV$. Then one just scales lengths and energies in opposite directions to maintain the result $\hbar c$. For instance, 1 Fermi=$10^{-15}m$ which is $10^{20} l_P$ which means that it corresponds to the energy $10^{-20} E_p = 100 MeV$ which is roughly the energy scale in nuclear physics (QCD). Recall that 1 Fermi is the size of the proton.

which is crazy. Try now two circular dimensions instead, i.e., set $D = 6$. Then

$$(l_P^{(6)})^4 = (l_P)^2\,(l_c)^2 \Rightarrow l_c = \frac{10^{-40}m}{10^{-35}m} = 10^{-5}m, \tag{2.43}$$

which is quite reasonable! Note that in this case, with $D = 6$, gravitational forces will behave as follows

$$F \sim r^{-2} \ \text{ for } \ r >> l_c, \tag{2.44}$$

$$F \sim r^{-4} \ \text{ for } \ r << l_c. \tag{2.45}$$

This is precisely where the limit for detectability of extra dimensions is today, see Hoyle et al . As we will discuss in detail later, one may define gravity to live in the whole spacetime (as done above) but define the Standard Model to live on a surface (D-brane) containing the non-compact four-dimensional spacetime that we normally regard as a our universe. This is called the *Brane world scenario*.

2.2 BZ chapter 4: Non-relativistic strings

It is now time to introduce strings which we first do in the simplest possible setting: Non-relativistic strings. These are physical in the same sense as a violin-string or the mesonic QCD-string that was mentioned above. This kind of string is the subject of this chapter but when we come actual string theory later the string is a generalisation of a point-particle and hence a fundamental object and not physical in the violin-string sense. For the fundamental string it is not possible to mark the points along the string since they are all identical, much like two electrons.

The string to be discussed here has its two ends fixed at $\mathbf{r_1} = (0,0,0)$ and $\mathbf{r_2} = (a,0,0)$. If left alone and without any gravitational field acting on it, its tension will force it to be a straight line in the x-direction between the two end points. We will then pull it out a small distance in the y-direction. When let go it will start oscillating with a small amplitude compared to its length, i.e., the amplitude y is a function of t, x with $y(t,x) << a$ for all $x \in [0,a]$ and all t.

The string has two basic properties: Tension, T_0, and mass/unit length, μ_0. Note that $[T_0] = [force] = N = \frac{Nm}{m} = \frac{energy}{length}$ and $[\mu_0] = \frac{kg}{m}$. For small oscillations we can consider the total length to be the same throughout the oscillations and hence the total mass to be given by $M = \mu_0 a$.

It is now rather straightforward to derive the differential equation that will govern these oscillations. Let us consider a small piece of the string, between x and $x + \Delta x$, where the amplitudes are $y(x)$ and $y(x) + \Delta y(x)$ (at some time t). Since we consider only small oscillations we have $\frac{\Delta y}{\Delta x} << 1$.

The force in the y-direction on the piece of the string between x and $x + \Delta x$ is

$$\Delta F_y(x) = F_y(x + \Delta x) - F_y(x). \tag{2.46}$$

The force comes from the pull on the end points from the rest of the string and is thus given by tension as follows (the tension acts along the tangent to the string)

$$\Delta F_y(x) = F_y(x+\Delta x) - F_y(x) = T_0 \frac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}|_{x+\Delta x} - T_0 \frac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}|_x. \tag{2.47}$$

Here Δy can be neglected in comparison to Δx which implies, letting $\Delta x \to dx \to 0$,

$$dF_y = T_0 \left(\frac{dy}{dx}|_{x+dx} - \frac{dy}{dx}|_x\right) \approx T_0 \frac{d^2 y(x)}{dx^2} dx. \tag{2.48}$$

The dynamics of the dx piece of the string at x is governed by Newton's second law which then becomes,

$$\mu_0 \, \ddot{y} = T_0 \, y''(x). \tag{2.49}$$

Here we have used $dF_y = dm\ddot{y}$, $dm = \mu_0 dx$ and represented an x-derivative by a prime (and used $df/dt := \dot{f}$). This equation is just the wave equation which then identifies the wave velocity as $v_0^2 = T_0/\mu_0$.

The wave equation above is second order in both time and x derivatives and hence needs two conditions for both t, *initial conditions*, and x, *boundary conditions*. Boundary conditions can be of two types. In the problem discussed above the two ends of the string are fixed (e.g., to a wall) which is written, for all t,

$$\text{Dirichlet b.c.: } y(t, x = 0) = 0, \;\; y(t, x = a) = 0. \tag{2.50}$$

If we had considered string ends that can slide without friction up and down (e.g., along a pole in the y-direction) then we would have

$$\text{Neumann b.c.: } y'(t, x = 0) = 0, \;\; y'(t, x = a) = 0. \tag{2.51}$$

These x-derivatives vanish since there are no forces (friction) from the pole in the y-direction so the sloop of the string must vanish (i.e., it must be parallell to the x-axis).

This discussion can be generalised in many ways some of which will become extremely important later:
1. Mixed boundary conditions: i.e, both D bc and N bc are used.
2. Adding a z-direction leads to the need for b.c. in both y and z directions and these can be the same or different. E.g., the sliding pole case discussed above would require N bc in the y-direction and D bc in the z direction for oscillations given by $(y(t, x), z(t, x))$.

Having introduced boundary conditions in the space directions the possible spectrum of modes can be determined. To see how these modes behave in time one has to solve the wave equation. To do this we need two initial conditions in time:

$$\text{Initial conditions : } \; y(t = 0, x) = f(x), \;\; \dot{y}(t = 0, x) = g(x). \tag{2.52}$$

Although the procedure outlined above for how to solve the wave equation is rather standard, the fact that the string corresponds to a two-dimensional problem in (t, x) implies that there is a more powerful way to proceed. We can start by solving the wave equation before introducing any initial or boundary conditions. To see that this is possible we rewrite the 2-dimensional wave equation using light-cone coordinates (with a slightly different definition from before):

$$x^{\pm} = x \pm v_0 t \Rightarrow \frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-} y(x^+, x^-). \tag{2.53}$$

Note that $\frac{\partial}{\partial x^+} x^+ = 1$ and $\frac{\partial}{\partial x^+} x^- = 0$ which imply

$$\frac{\partial}{\partial x^+} = \frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{v_0}\frac{\partial}{\partial t}), \;\; \frac{\partial}{\partial x^-} = \frac{1}{2}(\frac{\partial}{\partial x} - \frac{1}{v_0}\frac{\partial}{\partial t}). \tag{2.54}$$

This gives immediately the general solution of the wave equation in terms of one right-moving (h_+) wave and one left-moving one (h_-):

$$y(t,x) = h_+(x - v_0 t) + h_-(x + v_0 t) = h_+(x^-) + h_-(x^+). \tag{2.55}$$

An example of the above are oscillations in the form of "standing waves". They can be expressed as

$$y(t,x) = y(x)\sin(\omega t + \phi), \tag{2.56}$$

which can satisfy either D bc or N bc at the two ends $x = 0$ and $x = a$, including one of each. The initial conditions are given by the function $y(x)$. First we shift t so $\phi = 0$ at $t = 0$. Then for $t = 0$ we have $y(t = 0, x) = 0$ but the velocity is given by $\dot{y}(t,x)|_{(t=0,x)} = \omega y(x)$. Thus the string starts along the x-axis at $t = 0$ but with velocity given by $\omega y(x)$. Inserting this into the wave equation gives

$$y''(x) + \frac{\omega^2}{v_0^2} y(x) = 0. \tag{2.57}$$

To solve this equation we must specify the boundary conditions. The general solution contains both cos and sin modes. So

$$\text{D bc at both ends}: \;\; y_n(x) = A_n \sin\frac{n\pi x}{a}, \;\; n = 1, 2, ... \tag{2.58}$$

Note that the zero mode ($n = 0$) is not present!

Using Neumann bc at both ends instead we get

$$\text{N bc at both ends}: \;\; y_n(x) = B_n \cos\frac{n\pi x}{a}, \;\; n = 0, 1, 2, ... \tag{2.59}$$

In both cases the frequency spectrum is given by

$$\omega_n = \sqrt{\frac{T_0}{\mu_0}}\frac{n\pi}{a}, \;\; n = 1, 2, ... \tag{2.60}$$

while for the N bc case with $n = 0$ we get *rigid linear motion* with $\omega_0 = 0$ given by the **zero modes** a and b:

$$y_0(t,x) = at + b. \tag{2.61}$$

Exercises:
1. What happens if we start form an ansatz with a cos instead of a sin for the time dependence?
2. What happens if we use mixed boundary condition: D bc at $x = 0$ and N bc at $x = a$?

Comment: As already mentioned the string discussed above is not a fundamental string like the one we will analyse in string theory. The points on the string we considered above can be marked with a pen and can then be followed separately as a function of time. Such a string can also have longitudinal oscillations. This is not possible for a fundamental string: The points can not be marked and no longitudinal motion can occur. The only thing we can do when describing a fundamental string is to order the points with a parameter.

It is of course important to have a Lagrangian formulation of the wave equation derived above. Since it will turn out to be very convenient to compare string theory to the simpler case of a point particle we start by reviewing that case. The point particle action $S[x]$ and Lagrangian $L(x(t))$ read (between the initial time t_i and final time t_f of the motion)

$$S[x] = \int_{t_i}^{t_f} dt\, L(x(t)), \text{ where } L(x(t)) = \frac{1}{2}m\dot{x}^2 - V(x). \tag{2.62}$$

We get the equation of motion using Hamilton's principle, that is, from a variational principle: Require $\delta S[x] = 0$ under a variation of x by δx satisfying $\delta x(t = t_i) = 0$ and $\delta x(t = t_f) = 0$. Thus

$$\delta S[x] := \delta S[x + \delta x] - \delta S[x] = \int dt(m\dot{x}\delta\dot{x} - \frac{dV(x)}{dx}\delta x) = 0. \tag{2.63}$$

Since the time derivative and the variation commute $\delta\dot{x} = \frac{d(\delta x)}{dt}$ we can integrate by parts in time to find

$$\delta S[x] = \int dt(-m\ddot{x} - \frac{dV(x)}{dx})\delta x + \int dt\frac{d}{dt}(m\dot{x}\delta x) = 0. \tag{2.64}$$

The last term is a boundary term in time so it vanishes by definition of the Hamilton's principle. The first term is called the **bulk term** since the integral is over the whole parameter space (here only time) and must then vanish by itself: This gives Newton's 2nd law

$$m\ddot{x} = -\frac{dV(x)}{dx} := F. \tag{2.65}$$

This calculation is very important to understand in detail since it will be repeated below for the (violin) string. Then there is also a boundary term in a space direction which will produce boundary conditions like D bc and N bc discussed above. The fundamental string case is slightly more complicated but essentially the same.

Now we turn to the (violin) string case. Then, recalling the definition of the Lagrangian, we have

$$L = E_{kin} - E_{pot} := T - V \text{ where } \begin{cases} T = \int_0^a \frac{1}{2}(\mu_0 dx)\dot{y}^2, \\ \\ V = \int_0^a T_0\, dl. \end{cases} \tag{2.66}$$

The potential energy term arises from to the stretching of the string dl that is due to the oscillations. Thus $dl = \sqrt{dx^2 + dy^2} - dx = dx(\sqrt{1 + (\frac{dy}{dx})^2} - 1) \approx \frac{1}{2}(\frac{dy}{dx})^2 dx$ since $\frac{dy}{dx}$ is assumed small here. So the Lagrangian

$$L = \int_0^a (\frac{1}{2}\mu_0\dot{y}^2 - \frac{1}{2}T_0 y'^2)dx := \int_0^a dx\mathcal{L}(t, x). \tag{2.67}$$

Comment: If we express the action in terms of the Lagrangian density $\mathcal{L}$ as follows

$$S[y] = \int_{t_i}^{t_f} dt \int_0^a dx\, \mathcal{L}(y(t,x), \dot{y}(t,x), y'(t,x)), \tag{2.68}$$

we can view this string theory as a 1+1-dimensional field theory with $y(t,x)$ as the field in a spacetime with coordinates (t,x). This is often very convenient and will be used frequently in this course.

The next task is to use this action to obtain the equations of motion for the string (i.e., the 1+1 dimensional field equations) and boundary conditions. Thus we compute the variation $\delta S[y]$ and set it to zero: To first order in δy we get

$$\delta S[y] = \delta S[y+\delta y] - \delta S[y] \approx \int_{t_i}^{t_f} dt \int_0^a dx(\mu_0 \dot{y}\,\delta\dot{y} - T_0 y'\,\delta y') = 0. \tag{2.69}$$

This expression must be integrated by parts in both t and x directions to get the bulk term proportional to δy:

$$\delta S[y] = \int dt \int dx\,(-\mu_0\ddot{y} + T_0 y'')\delta y + \int dt \int dx\,(\partial_t(\mu_0\dot{y}\delta y) - \partial_x(T_0 y'\delta y)) = 0. \tag{2.70}$$

There are now two kinds of implications from this equation:

1) The bulk term=0,
2) The boundary terms =0.

Thus

$$\text{Bulk terms=0} \Rightarrow\ -\mu_0\ddot{y} + T_0 y'' = 0, \tag{2.71}$$

$$\text{Boundary terms=0} \Rightarrow\ y'\delta y|_{x=0} = 0, \text{ and } y'\delta y|_{x=a} = 0. \tag{2.72}$$

Here we have used that the boundary terms in the time direction vanish by definition. The first equation is just the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2} = 0, \tag{2.73}$$

while the two boundary equations can be satisfied in two different ways (for all t):

$$\text{Dirichlet bc:}\quad \delta y(t, x=0) = 0 \text{ or equivalently } \frac{\partial y}{\partial t}(t, x=0) = 0, \tag{2.74}$$

$$\text{Neumann bc:}\quad \frac{\partial y}{\partial x}(t, x=0) = 0, \tag{2.75}$$

and similarly for the (independent) boundary equation at $x = a$.

Comments: The total momentum of all the points on the string in the y-direction is $(dm = \mu_0 dx)$

$$p_y = \int_0^a dx \mu_0 \dot{y} \Rightarrow \frac{\partial p_y}{\partial t} = \mu_0 \int_0^a dx \ddot{y} = T_0 \int_0^a dx y'' = T_0 \frac{\partial y}{\partial x}\Big|_0^a = \begin{cases} = 0 \text{ for Neumann bc,} \\ \\ \neq 0 \text{ for Dirichlet bc.} \end{cases} \tag{2.76}$$

So while Neumann bc implies conservation of momentum (the string is free in the y-direction in this case) this is not the case for Dirichlet bc. But this is also OK since the momentum is absorbed by the wall to which the string ends are stuck.

Comments: We will also define the useful quantities (note the position of the indices)

$$\mathcal{P}^t := \frac{\partial \mathcal{L}}{\partial \dot{y}}, \quad \mathcal{P}^x := \frac{\partial \mathcal{L}}{\partial y'}. \tag{2.77}$$

These are easily obtained from the Lagrangian above:

$$\mathcal{P}^t := \mu_0 \, \dot{y}, \quad \mathcal{P}^x := -T_0 \, y'. \tag{2.78}$$

Using these quantities the equation of motion reads

$$\partial_t \mathcal{P}^t + \partial_x \mathcal{P}^x = 0, \tag{2.79}$$

which looks like the equation for a conserved current (more later).

3 Lecture 3

In this lecture we will take the first serious steps towards string theory. Many properties of the fundamental string are direct generalisations of the simpler object, the relativistic point particle, so this is good place to start.

3.1 BZ Chapter 5: The relativistic point particle

The motion in spacetime of a point particle with mass takes place inside the light-cone. Even if it has no velocity in space it traces out a **world-line** in Minkowski space $x^\mu(\tau)$ where τ is a (time-like) parameter along the path. If there are no forces acting on it, we say that it is **free**, and then it moves on straight lines inside the light-cone. Similarly, if the particle is massless, like the photon, it moves on the light-cone.

The classical equation that governs its motion can be obtained from Hamilton's principle (the variational principle) applied to the action

$$S = -mc\int_{p_1}^{p_2} ds = -mc\int_{\tau_1}^{\tau_2}\sqrt{-dx^\mu(\tau)dx^\nu(\tau)\eta_{\mu\nu}}, \tag{3.1}$$

which is thus given by the proper time interval ds ($ds^2 > 0$ or $ds^2 = 0$). As usually done in special relativity we can choose $\tau = t$ and extract a factor dt from the square root:

$$S = -mc\int_{t_1}^{t_2} dt\sqrt{-\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\eta_{\mu\nu}} = -mc^2\int dt\sqrt{1-\frac{\mathbf{v}^2}{c^2}}. \tag{3.2}$$

Let us now explain why this is the correct form of the action in this case. First we note that it is relativistically invariant. Secondly, it must give the correct non-relativistic result for small velocities which it does:

$$S \approx -mc^2\int dt(1-\frac{1}{2}\frac{\mathbf{v}^2}{c^2}+...) = -mc^2(t_2-t_1)+\frac{1}{2}mc^2\int dt\frac{\mathbf{v}^2}{c^2}+.. \tag{3.3}$$

Here we see that, since $L = T - V$, the rest mass mc^2 belongs to the potential energy V and $\frac{1}{2}mc^2\frac{\mathbf{v}^2}{c^2} = \frac{1}{2}m\mathbf{v}^2$ is just the kinetic energy T. This also shows that S has the correct dimension, namely the same dimension as $\hbar$: $Nms = kgm^2/s$. This means that it is dimensionless in natural units!

As a further check we may compute the canonical momentum:

$$\mathbf{p} := \frac{\partial L}{\partial\dot{\mathbf{r}}} = \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{3.4}$$

and the Hamiltonian

$$H := \mathbf{p}\cdot\dot{\mathbf{r}} - L = \frac{m\mathbf{v}^2}{\sqrt{1-\frac{v^2}{c^2}}} + mc^2\sqrt{1-\frac{v^2}{c^2}} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{3.5}$$

both coming out correctly!

We can now vary the action to obtain the equation of motion for the relativistic point particle (τ is a general parameter along the world-line while the time-like interval is denoted ds as above):

$$\delta x^\mu(\tau) \Rightarrow \delta S = -mc \int \delta(ds) = -mc \int \delta(\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}) = -mc \int \delta(\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}) d\tau. \tag{3.6}$$

Here we have introduced the velocity with respect to the τ-parameter : $\dot{x}^\mu(\tau) := \frac{dx^\mu}{d\tau}$. Note that this is not a Lorentz vector which instead is obtained using the interval s as the parameter, i.e., $u^\mu := \frac{dx^\mu}{ds}$.

To continue the variation of the action we write it as follows

$$\delta S = \int d\tau \frac{\partial L}{\partial \dot{x}^\mu} \delta \dot{x}^\mu. \tag{3.7}$$

Then the answer follows directly:

$$\delta S = mc \int d\tau \frac{\eta_{\mu\nu}\dot{x}^\mu \delta\dot{x}^\nu}{\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma}}. \tag{3.8}$$

We may now choose to continue this calculation using τ as the parameter but this will mean keeping square root expressions in various places. There is, however, a nice way to simplify the result by observing that

$$\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma} = \sqrt{(\frac{ds}{d\tau})^2} = \frac{ds}{d\tau}, \tag{3.9}$$

where in the last equality we use the fact that $\frac{ds}{d\tau} > 0$ since any parameter along the world-line must parametrise the path monotonically (i.e., uniquely). So using this fact together with

$$\dot{x}^\mu = \frac{dx^\mu(\tau)}{d\tau} = \frac{ds}{d\tau}\frac{dx^\mu(s)}{ds} = \frac{ds}{d\tau} u^\mu, \tag{3.10}$$

the variation of the action reads

$$\delta S = mc \int d\tau \frac{\eta_{\mu\nu}\frac{ds}{d\tau} u^\mu \delta\dot{x}^\nu}{\frac{ds}{d\tau}} = c \int \eta_{\mu\nu} p^\mu \delta\dot{x}^\nu d\tau, \tag{3.11}$$

which can be integrated by parts in τ to give

$$\delta S = -c \int \eta_{\mu\nu}\dot{p}^\mu \delta x^\nu + c[\eta_{\mu\nu} p^\mu \delta x^\nu]\Big|_{\tau_i}^{\tau_f} = 0. \tag{3.12}$$

The boundary terms are only in the time direction so they vanish by definition of Hamilton's principle. Hence only the bulk term is non-trivial:

$$\dot{p}^\mu = \frac{dp^\mu}{d\tau} = 0. \tag{3.13}$$

Note that this can be expressed in a Lorentz covariant way by using $ds = \sqrt{-\dot{x}^2}d\tau$ as

$$\frac{dp^\mu}{ds} = 0, \text{ or } m\frac{d^2x^\mu(s)}{ds^2} = 0. \tag{3.14}$$

Comments: The above variation of the point particle action can also be done (as in the gravity course) in a spacetime with a general metric $g_{\mu\nu}(x)$ replacing the Minkowski one $\eta_{\mu\nu}$. The result is then the *geodesic equation*

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}(g)\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0, \tag{3.15}$$

where $\Gamma^\mu_{\nu\rho}(g)$ is the affine connection constructed from the metric $g_{\mu\nu}$. If this equation is expressed in terms of a world-line parameter other than the invariant interval s, e.g., τ, a metric and a covariant derivative must be introduced also on the world-line by the replacement

$$\frac{d^2x^\mu}{ds^2} \to \nabla_\tau\partial_\tau x^\mu(\tau) := \Box_{(\tau)}x^\mu(\tau) = \frac{1}{\sqrt{-g}}\partial_\tau\left(\sqrt{-g}g^{\tau\tau}\partial_\tau x^\mu(\tau)\right). \tag{3.16}$$

exactly as if we had viewed $x^\mu(\tau)$ as a set of scalar fields living in a spacetime with only a time direction and metric $g_{\tau\tau}$ with "determinant" $\det g = g_{\tau\tau} < 0$.

We end this discussion of the point particle by giving it a charge q and couple it to a *background* electromagnetic field A_μ, i.e., this field is fixed and is not affected by the charged particle, i.e., we are neglecting the back-reaction. The rather simple definitions and formulas below are quite important since they will be generalised in several ways later in the context of the string. Recall first the Lorentz force law

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}). \tag{3.17}$$

The relativistic version reads, using the interval s as the parameter on the world-line,

$$\frac{dp_\mu}{ds} = \frac{q}{c}F_{\mu\nu}u^\nu. \tag{3.18}$$

The action that gives rise to this dynamical equation for the charged point particle is

$$S = -mc\int ds + \frac{q}{c}\int ds\frac{dx^\mu(s)}{ds}A_\mu(x(s)) = -mc\int_{\mathcal{P}} ds + \frac{q}{c}\int_{\mathcal{P}} A, \tag{3.19}$$

where we have formed the so called *1-form* $A := dx^\mu A_\mu(x)$ used heavily in differential geometry. $\mathcal{P}$ denotes the path of the particle in spacetime (i.e., just the world-line but without a specified parametrisation).

Comment: Note the important fact that although the field $A_\mu(x)$ is defined everywhere in spacetime $A_\mu(x(\tau))$ is defined only on the world-line.

Finally, the complete system of a charged particle moving in the background of a dynamical electromagnetic field must also contain Maxwell's equations. The complete action is then

$$S = -mc \int_{\mathcal{P}} ds + \frac{q}{c} \int_{\mathcal{P}} A - \frac{1}{4c} \int d^4x F_{\mu\nu} F^{\mu\nu}. \tag{3.20}$$

Note that this action contains terms which are integrated over different manifolds (spacetime and the world-line). This way of writing the action is nice since it takes (almost) exactly the same form (the F^2 term needs a $\sqrt{-g}$) if the geometry of spacetime is curved given by a general metric $g_{\mu\nu}$. If also spacetime is dynamic we need in addition Einstein's equations. This means adding also the Einstein-Hilbert term to the above action (all the other terms are already made generally covariant according to the rules in GR).

Note that the 1-form term $\int A := \int dx^\mu A_\mu$ is special: It takes the same form also in curved spacetimes; it is by definition coordinate independent as it is written (that is, without using the metric).

3.2 BZ Chapter 6: Relativistic strings

The main goal now is to generalise the results for the point particle found above to the case of the string. We will do this along the lines of some discussions in geometry that should familiar to most of you (they appear in the course Gravitation and Cosmology).

First we recall that the way to get a metric on the 2-sphere S^2 is simply to embed it in flat 3-space $\mathbf{R}^3$ by solving the definition equation of the embedding: $x^2 + y^2 + z^2 = R^2$. Introducing the angular coordinates θ, ϕ on S^2 we get standard result ($m = 1, 2, 3$ and $x^1 := x$ etc)

$$x^m(\theta,\phi): \quad x(\theta,\phi) = R\sin\theta\cos\phi,\ y(\theta,\phi) = R\sin\theta\sin\phi,\ z(\theta,\phi) = R\cos\theta. \qquad (3.21)$$

The metric on S^2 denoted $g_{ij}(\theta,\phi)$ ($i, j = (\theta,\phi)$) is obtained from the flat metric on $\mathbf{R^3}$ as follows:

$$ds^2(S^2) = (dx(\theta,\phi))^2 + (dy(\theta,\phi))^2 + (dz(\theta,\phi))^2 = g_{ij}(\theta,\phi)d\theta d\phi, \qquad (3.22)$$

where

$$g_{ij}(\theta,\phi) = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^m}{\partial\theta}\frac{\partial x^n}{\partial\theta} & \frac{\partial x^m}{\partial\theta}\frac{\partial x^n}{\partial\phi} \\ \frac{\partial x^m}{\partial\phi}\frac{\partial x^n}{\partial\theta} & \frac{\partial x^m}{\partial\phi}\frac{\partial x^n}{\partial\phi} \end{pmatrix} \delta_{mn}. \qquad (3.23)$$

If we now denote the angular coordinates as $\sigma^i = (\theta,\phi)$ the above equation reads simply

$$g_{ij}(\theta,\phi) = \frac{\partial x^m}{\partial\sigma^i}\frac{\partial x^n}{\partial\sigma^j}\delta_{mn}, \qquad (3.24)$$

which gives the crucial result that the metric $g_{ij}(\theta,\phi)$ on S^2 is the **pull-back** of the metric δ_{mn} on $\mathbf{R}^3$. This fundamental result can be generalised to any situation where the embedding of a *surface* $\Sigma_d(\sigma^i)$ into some **target manifold** $\mathcal{M}_D(x^m)$ is defined by maps $x^m(\sigma^i)$ from the surface to the target manifold. Here the dimensionalities of the surface and target space are given by d and D, respectively.

The volume (or area) element of the surface is as always in general relativity given by the determinant of the metric as

$$V = \int_{\Sigma_d} d^d\sigma\sqrt{det g_{ij}}. \qquad (3.25)$$

As a concrete example we consider a soap film extended between two rings held parallell to each other and a fixed distance apart. If we neglect gravity the film will intuitively be of cylindrical shape but with a smaller radius at the middle due to the tension in the film. The exact shape is obtained by solving the equations for the film derived by varying the embedding functions $x^m(\sigma^i)$ using Hamilton's principle as we did above.

This calculation will be done below but first we derive the soap film result in a second more direct way and then we bring it over to spacetime and relativistic string theory. The alternative derivation starts from the soap film between the two rings. Introduce now two arbitrary coordinates (but not parallell anywhere) spanning the surface at any point on it. As above we denote them as (σ_1, σ_2) and draw lines on the film corresponding to these coordinates. At each point (σ_1, σ_2) we can compute the tangent vectors to the coordinate lines by

$$v_1^m = \frac{\partial x^m}{\partial \sigma^1}, \; v_2^m = \frac{\partial x^m}{\partial \sigma^2}. \tag{3.26}$$

An infinitesimal area element on the film, with one corner at some given point, is then obtained from the cross product between these two tangent vectors $dA = |\mathbf{v}_1 \times \mathbf{v}_2| d\sigma^1 d\sigma^2$. Note that side 1 of the parallellogram is $(dx^m)_1 = v_1^m \, d\sigma_1$ and the same for the second one. However, this area element can be written $dA = |\mathbf{v}_1||\mathbf{v}_2| \sin\alpha \, d\sigma^1 d\sigma^2$ where α is the angle between the two tangent vectors. Then

$$dA = |\mathbf{v}_1||\mathbf{v}_2|\sqrt{1-\cos^2\alpha} \, d\sigma^1 d\sigma^2 = \sqrt{|\mathbf{v}_1|^2|\mathbf{v}_2|^2 - |\mathbf{v}_1|^2|\mathbf{v}_2|^2\cos^2\alpha} \, d\sigma^1 d\sigma^2. \tag{3.27}$$

Inserting the expressions for the tangent vectors above this becomes

$$dA = \sqrt{(\frac{\partial x^m}{d\sigma^1}\frac{\partial x^n}{d\sigma^1}\delta_{mn})(\frac{\partial x^p}{d\sigma^2}\frac{\partial x^q}{d\sigma^2}\delta_{pq}) - (\frac{\partial x^m}{d\sigma^1}\frac{\partial x^n}{d\sigma^2}\delta_{mn})^2} d\sigma^1\sigma^2 = \sqrt{det g_{ij}} d\sigma^1 d\sigma^2, \tag{3.28}$$

since $\frac{\partial x^m}{d\sigma^1}\frac{\partial x^n}{d\sigma^2}\delta_{mn} = \mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2|\cos\alpha$. This is what we set out to prove.

The equation we are after for the string then follows directly from the above equations by replacing the surface of the soap film by the **world-sheet**, generated when the string moves in spacetime. Thus the coordinates on the world-sheet must have one coordinate with a tangent vector inside the light-cone and one space-like one. These are now denoted $\xi^\alpha = (\tau, \sigma)$ and the embedding functions $X^\mu(\tau, \sigma)$, called **string coordinates**, are telling us where in spacetime the world-sheet is:

$$X^\mu(\xi^\alpha): \;\; \Sigma_2 \to M_D. \tag{3.29}$$

Here Σ_2 is the world-sheet and M_D the D-dimensional **target space** which is just Minkowski space but could be any kind of spacetime even ones containing black holes etc. Note the new convention to use a capital letter for the string coordinates $X^\mu(\tau, \sigma)$.

This string has tension and hence an energy density along the string. This implies that any finite section of the string has a finite mass and must move with a velocity less than the speed of light. Contrary to a string that is not fundamental, our fundamental one does not have some atom being the last one at the ends of it. Hence one can consider the motion of smaller and smaller pieces close to a string end. Since the mass of such a piece of string will also go to zero as the length of the piece goes to zero, the end must always move with the speed of light! This is the first real clear (see below) new feature we encounter valid for a fundamental string.

The string action functional is called the **Nambu-Goto action** and reads

$$S[X^\mu] = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{-det\gamma_{\alpha\beta}}, \tag{3.30}$$

where we in this relativistic case, with world-sheet coordinates $\xi^\alpha = (\tau, \sigma)$, denote the pull-back metric on the world-sheet as $\gamma_{\alpha\beta}$, i.e.,[7]

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta}\eta_{\mu\nu}. \tag{3.31}$$

The above normalisation of the action follows from the fact that the tension T_0 has dimension $[T_0] = [force] = ML/T^2$ and $[S] = [\hbar] = [time \times energy] = ML^2/T$ which then implies the factor of c in the denominator. Note that the integrand is dimensionless in natural units but has general units as $[c]$ since coordinates in spacetime have dimension L (there is a factor c in the X^0 inside the $det\gamma_{\alpha\beta}$) while $\xi^0 := \tau$.

Explicitly the Nambu-Goto integrand reads

$$\sqrt{-det\gamma_{\alpha\beta}} = \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}. \tag{3.32}$$

The computation of the Nambu-Goto string equations of motion will be done along the lines discussed above. Thus we have

$$\delta X^\mu(\xi^\alpha) \Rightarrow \delta S[X^\mu] = \int d\tau d\sigma \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\delta\dot{X}^\mu + \frac{\partial \mathcal{L}}{\partial X'^\mu}\delta X'^\mu\right), \tag{3.33}$$

where dots are τ derivatives and primes σ derivatives. To avoid writing out complicated expressions containing squared roots we keep the notation here and continue the computation by performing integrations by parts in the standard way in both τ and σ:

$$\delta S[X^\mu] = -\int d\tau d\sigma \left(\partial_\tau(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}) + \partial_\sigma(\frac{\partial \mathcal{L}}{\partial X'^\mu})\right)\delta X^\mu \tag{3.34}$$

$$+\int d\tau d\sigma \left(\partial_\tau(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\delta X^\mu) + \partial_\sigma(\frac{\partial \mathcal{L}}{\partial X'^\mu}\delta X^\mu)\right) = 0. \tag{3.35}$$

To satisfy this equation we set the bulk term (first line) and the boundary terms (second line) to zero separately. This implies

$$\text{bulk term=0}: \ \partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0, \text{ where } \mathcal{P}^\tau_\mu := \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \ \mathcal{P}^\sigma_\mu := \frac{\partial \mathcal{L}}{\partial X'^\mu}. \tag{3.36}$$

Here we should note the position of the two indices on the quantities $(\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu)$. Note also that the equation of motion $\partial_\alpha \mathcal{P}^\alpha_\mu = 0$ looks like a two-dimensional current conservation equation as obtained from Noether's theorem (more later).

Turning to the boundary terms we note first that the first term on the second line is zero by definition (boundary terms in the τ direction) while the second term gives

$$\text{space-like boundary terms =0: } (\mathcal{P}^\sigma_\mu \delta X^\mu)\Big|_{\sigma=0} = (\mathcal{P}^\sigma_\mu \delta X^\mu)\Big|_{\sigma=\sigma_1} = 0. \tag{3.37}$$

[7]In differential geometry this pull-back is written $\gamma = X^*(\eta)$.

Again we find that there are two different ways to satisfy these equations: Making a 1+3 split of the spacetime directions according to $\mu = (0, i)$ gives, **at each end** of the string and **in each space direction** i, one of two independent choices (for all τ)

$$\mu = i \begin{cases} \text{Free end bc (later Neumann): } \mathcal{P}^\sigma_i = 0, \\ \text{Dirichlet bc: } \delta X^i = 0 \text{ or equivalently } \frac{\partial X^i}{\partial \tau} = 0 \end{cases} \tag{3.38}$$

The many possible combinations of these boundary conditions will be studied in detail below. In the time direction $\mu = 0$ only the free condition is possible (time can not stop), i.e. at each end of the string we must impose

$$\mu = 0: \;\; \text{Free end bc (later Neumann): } \mathcal{P}^\sigma_0 = 0. \tag{3.39}$$

Before discussing the various kinds of boundary condition we should write down $(\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu)$ explicitly to see that they are quite complicated and that any kind of simplifications we can find would be very useful. They read

$$\mathcal{P}^\tau_\mu := \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')X'_\mu - X'^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}}, \tag{3.40}$$

$$\mathcal{P}^\sigma_\mu := \frac{\partial \mathcal{L}}{\partial X'^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\dot{X}_\mu - \dot{X}^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}}. \tag{3.41}$$

Returning to the boundary conditions we will introduce a table which provides a very convenient way of stating them: (*end 1* has $\sigma = 0$ and *end 2* $\sigma = \sigma_1$ where σ_1 is just a number at this point which will be determined later)

direction end:	1	2
x-direction	N bc	N bc
y-direction	D bc	N bc
z-direction	N bc	N bc

In this example the string lives in three space dimensions with the $\sigma = 0$ end stuck to a wall at $y = y_0$ and can hence move freely in the x and z directions. The other end has N bc in all three directions and is therefore free to move in all space directions. We will discuss many situations like this so we will introduce the extremely important concept of ***Dp*-branes**.

Dp-branes are not fundamental objects like the string we study here (which generalises the point-particles discussed previously[8]) but should instead be viewed as physical objects (like a chair or a black hole build from the fundamental objects in the theory). In the above example the $\sigma = 0$ end of the string is attached to a $D2$-brane parallell to the xz-plane

[8]These can also be viewed as the excitations created by $a^\dagger$ operators in QFT. This point of view leads in string theory to what is called *string field theory*.

while the $\sigma = \sigma_1$ end can be considered as stuck to a $D3$-brane. Since this last brane fills all of space it is called **space-filling**.

Comments: Dp-branes are
1) Physical objects that extend indefinitely in all its internal directions and are hence infinitely heavy.
2) They can also be of finite size (like spheres) but this is rather difficult to analyse[9].
3) If some dimensions are made compact Dp-branes may have no directions in the uncompactified directions and will then look like a point in these directions (e.g., those of our ordinary four-dimensional spacetime).

The static gauge: We will end this chapter by making use of the reparametrisation invariance on the world-sheet, i.e., in the coordinates (τ, σ) to simplify the complicated expressions for $(\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu)$ given above. This will be done in three steps of which the first one is taken here. The remaining two steps will be described in the next lecture.

Step 1: The static gauge is using the fact that, in a given Lorentz system in target space with coordinates x^μ, we may at time x^0 take a snap-shop of the string when it moves in space. On the world-sheet this string corresponds to a line from one side to the other, a line that we can then associate with a fixed value of the τ coordinate by setting it equal to the value of x^0. Thus we have chosen the so called static gauge: If we let Q denote any point on the snap-shot line then $\tau(Q) = t$ where $x^0 := ct$. We write this as a condition on the time component of the string coordinates as

$$\text{The static gauge: } X^0(\tau, \sigma) := cT(\tau, \sigma) = c\tau. \tag{3.42}$$

The static gauge therefore says that the string coordinate component $X^0(\tau, \sigma)/c$ and the world-sheet time coordinate τ are both equal to the coordinate time t (in the Lorentz frame chosen).
This gauge choice has some quite nice consequences: (using the 1+3 index split $\mu = (0, i)$)

$$X^\mu(\tau, \sigma) = X^\mu(t, \sigma) = (ct, X^i(t, \sigma)), \tag{3.43}$$

which implies

$$\dot{X}^\mu(\tau, \sigma) := \partial_\tau X^\mu = (\partial_t X^0, \partial_t X^i) = (c, \mathbf{v}), \tag{3.44}$$

$$X'^\mu(\tau, \sigma) := \partial_\sigma X^\mu = (\partial_\sigma X^0, \partial_\sigma X^i) = (0, \partial_\sigma \mathbf{r}). \tag{3.45}$$

In this gauge it is clear that the expression under the square root, that is $-det\gamma_{\alpha\beta}$, is positive! (Check this! There is an argument in the book valued more generally.)

We will end this chapter by discussing some of the physics of the string that can be made clear using the static gauge. So consider a fixed string in d space dimensions stretched

[9] See, e.g., "Spherical branes", arXiv: hep-th/1805.05338.

(and at rest) in the x_1 direction between $(0, ..., 0)$ and $(a, 0, ..., 0)$ where the former is the $\sigma = 0$ end and the latter the $\sigma = \sigma_1$ end. To evaluate the string action we note that

$$X^\mu(\tau,\sigma) = (X^0(\tau,\sigma), X^1(\tau,\sigma), X^I(\tau,\sigma)) = (c\tau, f(\sigma), 0, ..., 0), \;\; I = 2, 3, ... \tag{3.46}$$

where $X^1(\tau,\sigma) = f(\sigma)$ with $f(\sigma)$ an arbitrary (monotonous) function along the string such that $f(0) = 0$ and $f(\sigma_1) = a$. This $X^\mu(\tau,\sigma)$ gives

$$\dot{X}^\mu = (c, 0, ..., 0), \;\; X'^\mu = (0, f'(\sigma), ..., 0), \;\; \text{where } \; f'(\sigma) > 0. \tag{3.47}$$

Using this result we get

$$\dot{X}^\mu \cdot X'_\mu = 0, \;\; (\dot{X}^\mu)^2 = -c^2, \;\; (X'^\mu)^2 = (f')^2, \tag{3.48}$$

which when inserted into the action gives

$$S[X^\mu] = -\frac{T_0}{c}\int d\tau d\sigma \sqrt{c^2(f')^2} = -T_0 \int d\tau d\sigma \, f'(\sigma) = -T_0 \int_{\tau_i}^{\tau_f} dt \int_0^{\sigma_1} d\sigma \partial_\sigma f(\sigma), \tag{3.49}$$

and hence since there is no time dependence in the integrand

$$S[X^\mu] = -T_0(t_f - t_i)(f(\sigma_1) - f(0)) = -T_0(\,t_f - t_i)\,a := \int dtL = \int dt(E_{kin} - E_{pot}). \tag{3.50}$$

The last equality leads to the identification (V is the potential energy and M the total rest mass)

$$E_{pot} := V = T_0\, a = Mc^2 \;\Rightarrow\; V/a = \mu_0 c^2 \;\Rightarrow\; \mu_0 c^2 = T_0. \tag{3.51}$$

For the above discussion to make sense at all the stretched string at rest must solve the equations of motion and satisfy sensible boundary condition. However, the bc are Dirichlet at both ends which are always correct (the wall compensates for all non-conserved quantities etc). To check that the equations of motion, i.e., the Nambu-Goto (NG) equations, $\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0$ are satisfied we need the two quantities $\mathcal{P}^\alpha_\mu$ computed from the $X^\mu(\tau,\sigma)$ above.

Since there is no τ dependence at all in this situation the NG-equations give just $\partial_\sigma \mathcal{P}^\sigma_\mu = 0$. Is this correct? We have from before

$$\mathcal{P}^\sigma_\mu = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - \dot{X}^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}} = -\frac{T_0}{c}\left(\frac{c^2 X'_\mu}{f'}\right) = -T_0 c(0, 1, 0..., 0), \tag{3.52}$$

which is constant and thus shows that the NG equations are satisfied.

It is in fact quite interesting to compute the general form of the action in the static gauge. To do this we need the expressions above for the string coordinates in the static gauge $\tau = t$:

$$X^\mu(\tau,\sigma) = X^\mu(t,\sigma) = (ct, X^i(ct,\sigma)), \tag{3.53}$$

which implies

$$\dot{X}^\mu(\tau,\sigma) = \partial_\tau X^\mu = (\partial_t X^0, \partial_t X^i) = (c, \mathbf{v}), \tag{3.54}$$

$$X'^\mu(\tau,\sigma) = \partial_\sigma X^\mu = (\partial_\sigma X^0, \partial_\sigma X^i) = (0, \partial_\sigma \mathbf{r}). \tag{3.55}$$

Computing the various Lorentzian scalar products gives

$$\dot{X}^2 = -c^2 + \dot{\mathbf{r}}^2, \tag{3.56}$$

$$X'^2 = \mathbf{r}'^2, \tag{3.57}$$

$$\dot{X} \cdot X' = \dot{\mathbf{r}} \cdot \mathbf{r}'. \tag{3.58}$$

Inserted into the expression under the square root, i.e., $-det\gamma_{\alpha\beta}$, it becomes

$$-\,det\gamma_{\alpha\beta} = (\dot{X} \cdot X')^2 - (\dot{X}^2)(X'^2) = (\dot{\mathbf{r}} \cdot \mathbf{r}')^2 + (c^2 - \dot{\mathbf{r}}^2)(\mathbf{r}'^2). \tag{3.59}$$

In order to clarify the meaning of this expression we introduce the *proper space length* denoted $d\bar{s}(\sigma)$ as follows (ignoring the τ dependence for now)

$$d\bar{s}(\sigma) := \sqrt{dx^i dx^i} = \sqrt{d\mathbf{r} \cdot d\mathbf{r}} = |d\mathbf{r}(\sigma)| = |\partial_\sigma \mathbf{r}| d\sigma = |\mathbf{r}'| d\sigma. \tag{3.60}$$

This implies some relations familiar from general relativity

$$(\mathbf{r}')^2 = (\frac{d\bar{s}}{d\sigma})^2, \quad \frac{d\mathbf{r}}{d\bar{s}} \cdot \frac{d\mathbf{r}}{d\bar{s}} = 1. \tag{3.61}$$

Then since $\mathbf{r}' = \partial_\sigma \mathbf{r} = \frac{d\bar{s}}{d\sigma}\, \partial_{\bar{s}} \mathbf{r}$ we have

$$-\,det\gamma_{\alpha\beta} = (\dot{\mathbf{r}} \cdot \mathbf{r}')^2 + (c^2 - \dot{\mathbf{r}}^2)(\mathbf{r}'^2) = (\frac{d\bar{s}}{d\sigma})^2 \left((\partial_t \mathbf{r} \cdot \partial_{\bar{s}} \mathbf{r})^2 + (c^2 - (\partial_t \mathbf{r})^2)(\partial_{\bar{s}} \mathbf{r})^2\right). \tag{3.62}$$

Using one of the equations above this becomes

$$-\,det\gamma_{\alpha\beta} = (\frac{d\bar{s}}{d\sigma})^2 \left((\partial_t \mathbf{r} \cdot \partial_{\bar{s}} \mathbf{r})^2 + (c^2 - (\partial_t \mathbf{r})^2)\right) = (\frac{d\bar{s}}{d\sigma})^2 \left(c^2 - ((\partial_t \mathbf{r})^2 - (\partial_t \mathbf{r} \cdot \partial_{\bar{s}} \mathbf{r})^2)\right). \tag{3.63}$$

Here we note the important fact that the expression in the last (inner) bracket is the square of the *velocity perpendicular to the string* denoted $\mathbf{v}_\perp$ (Check this!):

$$\mathbf{v}_\perp := \partial_t \mathbf{r} - (\partial_t \mathbf{r} \cdot \partial_{\bar{s}} \mathbf{r}) \partial_{\bar{s}} \mathbf{r}. \tag{3.64}$$

Thus we get

$$-\,det\gamma_{\alpha\beta} = (\frac{d\bar{s}}{d\sigma})^2 c^2 (1 - \frac{\mathbf{v}_\perp^2}{c^2}). \tag{3.65}$$

So finally we see that in the static gauge the action can be written

$$S = T_0 \int dt \int_0^{\sigma_1} d\sigma \frac{d\bar{s}}{d\sigma} \sqrt{1 - \frac{\mathbf{v}_\perp^2}{c^2}} = T_0 \int dt \int_0^{\sigma_1} d\bar{s} \sqrt{1 - \frac{\mathbf{v}_\perp^2}{c^2}}, \tag{3.66}$$

where the last expression emphasises the reparametrisation invariance along the string.

The importance of the above way of writing the string action (in the static gauge) comes from the fact that only the *velocity perpendicular to the string* denoted $\mathbf{v}_\perp$ appears in it. Thus any motion of the string parallell to the string is unobservable. This implies that it is not possible to mark a point on the string and follow its motion in time: Points on the string are not distinguishable (but possible to put in an order along the string)!

We can now draw two more conclusions about the general motion of the ends of an open string with free boundary conditions:
–They move with the speed of light.
–They move perpendicular to the string.

We now show explicitly that the facts follow free the boundary conditions

$$\mathcal{P}^\sigma_\mu(\tau,\bar\sigma)=0 \text{ for } \bar\sigma=0, \text{ and/or } \bar\sigma=\sigma_1. \tag{3.67}$$

Start by writing these bc in the static gauge. They read (see formulas for $\dot{X}^\mu$ etc above)

$$\text{In space directions}\,(\mu=i):\ \mathcal{P}^\sigma_i=-\frac{T_0}{c}\frac{(\dot{\mathbf{r}}\cdot\mathbf{r}')\dot{X}_i+(c^2-\mathbf{v}^2))X'_i}{(\frac{d\bar s}{d\sigma})c\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}=0, \tag{3.68}$$

$$\text{In the time directions}(\mu=0):\ \mathcal{P}^\sigma_0=-\frac{T_0}{c}\frac{(\dot{\mathbf{r}}\cdot\mathbf{r}')}{(\frac{d\bar s}{d\sigma})c\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}=0, \tag{3.69}$$

where we in the last equation have used the static gauge results $\dot{X}^0=1$ and $X'^0=0$. Thus the last equation implies

$$\dot{\mathbf{r}}\cdot\mathbf{r}'=0, \tag{3.70}$$

which means that the end of the string moves perpendicular to the string (note that $\mathbf{r}'$ is tangent to the string).

Inserting this result in the equation for the free bc in the space directions gives

$$\text{In space directions}\,(\mu=i):\ \mathcal{P}^\sigma_i=-T_0\frac{(1-\frac{\mathbf{v}^2}{c^2}))X'_i}{(\frac{d\bar s}{d\sigma})\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}=-T_0\sqrt{1-\frac{\mathbf{v}^2}{c^2}}\partial_{\bar s}X_i=0, \tag{3.71}$$

which implies that $v=c$, i.e., the end point moves with the speed of light.

4 Lecture 4

In this lecture we will take the two last steps of gauge fixing by using the remaining reparametrisation invariance of the world-sheet. We then obtain the true physical content of the string, that is, its degrees of freedom etc. The end result is, however, still not in the best possible form which will force us to use light-cone techniques. This will be discussed in the next lecture. The rest of this lecture will instead be devoted to symmetries and conserved currents defined on the world-sheet.

4.1 BZ Chapter 7: String reparametrisation invariance and classical motion

In the static gauge discussion in the previous lecture we mentioned that this gauge choice is just the first of three possible ones. So to simplify the string action and equations of motion in the Nambu-Goto formulation as far as possible we should also make use of the remaining reparametrisation invariance. (We will later rewrite the Nambu-Goto theory in a much better way using the so called Polyakov formulation.)

So, in Step 1 we used the reparametrisation invariance in the τ coordinate on the world-sheet to choose the **static gauge**

$$\text{Step 1: The static gauge:} \quad X^0(\tau,\sigma) = c\tau = ct. \tag{4.1}$$

Turning to the σ coordinate on the world-sheet we can use its reparametrisation invariance as follows: Start from the string at $t_0 = 0$ (i.e., the snap-shop at $t = 0$ in a given Lorentz system) and denote the coordinate along the string as σ_0. Then consider the snap-shot string at a small time $t_1 = dt$ later. If we on that string choose the σ coordinate such that at each point orthogonally after the corresponding point on the t_0-string is also given the value σ_0 then the τ-coordinate lines will be perpendicular to the σ-coordinate lines *everywhere* on the world-sheet not just at the ends (as we saw in the previous lecture). This is expressed as (with the 1+3 index split $\mu = (0, i)$):

$$\text{Step 2: The orthogonal gauge:} \quad \partial_\sigma X^i \partial_\tau X^i = 0 \quad \text{for all points on the world-sheet.} \tag{4.2}$$

This gauge choice has some very nice consequences for the string coordinates X^μ and momenta $\mathcal{P}^\alpha_\mu$. However, first we note that after choosing these two first gauges the string automatically moves perpendicular to itself, i.e.,

$$\mathbf{v} = \mathbf{v}_\perp. \tag{4.3}$$

We also see that the various Lorentz contractions now read

$$\dot{X} \cdot X' = 0, \tag{4.4}$$

$$\dot{X}^2 = -c^2(1 - \frac{v^2}{c^2}), \tag{4.5}$$

$$X'^2 = (\frac{d\bar{s}}{d\sigma})^2. \tag{4.6}$$

This implies

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}(\frac{d\bar{s}}{d\sigma})\frac{\dot{X}^\mu}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{4.7}$$

$$\mathcal{P}^{\sigma\mu} = -T_0\sqrt{1-\frac{v^2}{c^2}}\partial_{\bar{s}}X^\mu. \tag{4.8}$$

Thus, in this gauge setting $\mu = 0$ gives $\mathcal{P}^{\sigma 0} = 0$ for all (τ, σ).

At this point it is important to note that all reparametrisation freedom on the world-sheet has been used *except* the one involving σ at $t = 0$. To see how to make use of this fact we return to the Nambu-Goto equations of motion:

$$\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0. \tag{4.9}$$

In the above gauge (after steps 1 and 2) we get, using that $\mathcal{P}^\sigma_0 = 0$,

$$\mu = 0: \;\; \partial_t \mathcal{P}^\tau_0 = 0. \tag{4.10}$$

From the result above we see that

$$\mathcal{P}^\tau_0 = \frac{T_0}{c}(\frac{d\bar{s}}{d\sigma})\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4.11}$$

If we recall that the tension is related to the mass density μ_0 along the string by $T_0 = \mu_0 c^2$ we see that $T_0 d\bar{s}$ is the energy in the piece of the string of proper length $d\bar{s}$. The equation above then implies that the relativistic energy $T_0 d\bar{s}/\sqrt{1-\frac{v^2}{c^2}}$ is constant in time ($d\sigma$ is time-independent by definition). One can then define the total energy of the string by the integral

$$E = \int \frac{T_0 d\bar{s}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{4.12}$$

We now turn to the space components of the Nambu-Goto equations

$$\mu = i: \;\; \partial_\tau \mathcal{P}^\tau_i + \partial_\sigma \mathcal{P}^\sigma_i = 0. \tag{4.13}$$

Inserting the above expressions for the currents this equation becomes

$$\partial_t \left(\frac{T_0}{c^2}(\frac{d\bar{s}}{d\sigma})\frac{\partial_t X^i}{\sqrt{1-\frac{v^2}{c^2}}} \right) = \partial_\sigma \left(T_0\sqrt{1-\frac{v^2}{c^2}}\partial_{\bar{s}}X^i \right). \tag{4.14}$$

By using the fact found above that the relativistic energy in a segment $d\bar{s}$ of the string is constant in time, we can write this equation as follows

$$\left(\frac{T_0}{c^2}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \right) \partial_t^2 X^i = \partial_{\bar{s}} \left(T_0\sqrt{1-\frac{v^2}{c^2}}\partial_{\bar{s}}X^i \right). \tag{4.15}$$

T_0 drops out so this equation can be written

$$\frac{1}{c^2}\partial_t^2 X^i = \frac{\sqrt{1-\frac{v^2}{c^2}}}{\frac{d\bar{s}}{d\sigma}}\partial_\sigma\left(\frac{\sqrt{1-\frac{v^2}{c^2}}}{\frac{d\bar{s}}{d\sigma}}\partial_\sigma X^i\right), \tag{4.16}$$

where the derivatives on the right hand side are now both with respect to σ, not $\bar{s}$.

This is good place to introduce the third and last reparametrisation gauge. Since all points on the world-sheet are labelled by (τ, σ) coordinates that are given values such that in Step 1: $\tau = t$ and in Step 2: σ for all $t \neq 0$ snap-shop strings have the same values as the one at $t = 0$ using the orthogonality condition we are left with the possibility of choosing how to exactly parametrise the $t = 0$ string. Thus we choose this σ coordinate so that the following condition is satisfied:

$$\text{Step 3: Energy condition: Choose } \; d\sigma \; \text{ so that } \; 1 = \frac{\frac{d\bar{s}}{d\sigma}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{4.17}$$

Note that $\partial_t \mathcal{P}_0^\tau = 0$ implies that the right hand side is time independent (which means that this condition is valid for all t along the world-sheet). Note also that the relation to energy mentioned above, i.e., the energy $dE = T_0 d\bar{s}/\sqrt{1-\frac{v^2}{c^2}}$, means that the condition in Step 3 can be written

$$d\sigma = \frac{dE}{T_0}, \text{ which implies } \sigma \in \; [0, \sigma_1] = [0, \frac{E}{T_0}], \tag{4.18}$$

where E is the total (constant) energy of the string.
We have now completely specified the coordinates (τ, σ) on the world-sheet so it is time to summarise the results of Steps 1, 2 and 3:

$$\text{Static gauge}: \;\; X^0(\tau, \sigma) = c\tau \;\; (\text{follows from } \; t = \tau), \tag{4.19}$$

$$\text{Orthogonality gauge: } \partial_t X^i(\tau, \sigma)\partial_\sigma X^i(\tau, \sigma) = 0, \tag{4.20}$$

$$\text{Energy gauge}: \;\; (\partial_\sigma X^i)^2 + \frac{1}{c^2}(\partial_t X^i)^2 = 1, \tag{4.21}$$

which imply

$$\text{Wave equation: } \partial_\sigma^2 X^i - \frac{1}{c^2}\partial_t^2 X^i = 0, \tag{4.22}$$

$$\text{Canonical momenta: } \mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\dot{X}^\mu, \;\; \mathcal{P}^{\sigma\mu} = -T_0 X'^\mu. \tag{4.23}$$

Finally we see that the *free boundary conditions* $\mathcal{P}^{\sigma\mu} = 0$ have become Neumann $\partial_\sigma X^\mu = 0$. Note also that the equations are not Lorentz covariant.

One reason for going through these steps of gauge choices is to show that they lead to enormous simplifications of the string equations but that we still have problems. These problems appear since we end up with conditions that are bilinear (quadratic) in the string coordinates, namely the orthogonality and energy conditions. In order to quantise the string we must be able to solve it completely and find all independent degrees of freedom that need to be turned into operators (as in QFT). This cannot be done if there are quadratic constraints on the string coordinates (viewing $X^\mu(\tau,\sigma)$ as quantum fields in a 2-dimensional Minkowski spacetime). So after having discussed some classical aspects of this problem below we will in the following lectures use **light-cone coordinates in Minkowski space** which will eliminate the problem associated with the quadratic constraints.

Stringy motion: We will now first discuss the motion of the open string and then look at the closed string.

For either the open or closed string we can easily solve the equation of motion which is just the free wave equation. As we have already discussed in the beginning of the course the best way to do this is to use light-cone coordinates on the world-sheet. Thus define $\sigma^\pm := ct \pm \sigma$ in terms of which the derivatives (defined to give $\partial_\pm \sigma^\pm = 1$) and wave equation read

$$\partial_\pm = \frac{1}{2}(\frac{1}{c}\partial_t \pm \partial_\sigma), \;\; \partial_+ \partial_- X^i(\sigma^+, \sigma^-) = 0. \tag{4.24}$$

The wave equation is now solved in full generality by

$$X^i(\tau, \sigma) = \frac{1}{2}(F^i(\sigma^+) + G^i(\sigma^-)), \tag{4.25}$$

where all the F^i and G^i are arbitrary functions.

For the open string we can, e.g., impose Neumann bc at both ends, written as (N, N)-bc. Then using

$$\partial_\sigma X^i(\tau, \sigma) = \frac{1}{2}(F'^i(\sigma^+) - G'^i(\sigma^-)), \tag{4.26}$$

we see that the N bc at

$$\sigma = 0 : \Rightarrow F'^i(u) = G'^i(u), \tag{4.27}$$

where prime indicates a derivative w.r.t. its argument u. Integrating this equation gives $G^i(u) = F^i(u) + a^i$ where a^i are constants which can be absorbed into $F^i(u)$ since it is anyway an arbitrary set of functions. We thus find that

$$X^i(\tau, \sigma) = \frac{1}{2}(F^i(\sigma^+) + F^i(\sigma^-)). \tag{4.28}$$

Turning to the N bc at the other end (at $\sigma_1 = E/T_0$) we get

$$\partial_\sigma X^i(\tau, \sigma)|_{\sigma=\sigma_1} = \frac{1}{2}(F'^i(ct + \sigma_1) - F'^i(ct - \sigma_1)) = 0. \tag{4.29}$$

Setting $u := ct - \sigma_1$ gives then the $2\sigma_1$ periodicity condition

$$F'^i(u + 2\sigma_1) = F'^i(u). \tag{4.30}$$

So we have arrived at the following important conclusion: $F^i(u)$ is a **quasi-periodic** set of functions:

$$F^i(u+2\sigma_1) = F^i(u) + 2\sigma_1 \frac{v_0^i}{c}, \tag{4.31}$$

where the constant term on the RHS has been designed so that v_0^i becomes the average velocity over one period $t \to t + \frac{2\sigma_1}{c}$.

We must now also solve the quadratic conditions coming from the orthogonality and energy gauge conditions. First we combine them into the following

$$\text{Reparametrisation constraint: } (\partial_\sigma X^i \pm \frac{1}{c}\partial_t X^i)^2 = 1. \tag{4.32}$$

The presence of the $\pm$ in this equation makes it equivalent to the above form of the two gauge conditions. Plugging in the form of the string coordinates we found above gives the quadratic constraints on F^i:

$$(\partial_\pm F^i(ct \pm \sigma))^2 = 1, \tag{4.33}$$

This equation can not be solved exactly without obtaining square roots. This will make it impossible to quantise each degree of freedom independently (as normally done in perturbation theory in QFT).

In order to see what this quadratic constraint means we can start guessing what kind of motions are possible. One such is the **rigid motion of the open string** which, if taking place in the x, y plane, can be expressed as

$$X^i(t,\sigma) = \frac{\sigma_1}{\pi} \cos\frac{\pi\sigma}{\sigma_1}(\cos\frac{\pi ct}{\sigma_1}, \sin\frac{\pi ct}{\sigma_1}). \tag{4.34}$$

That this rigid motion solves all the equations can be seen as follows. The $\sigma = 0$ end of the string moves on a circle according to, using $\sigma_1 = \frac{E}{T_0}$,

$$X^i(t,0) = \frac{l}{2}(\cos\omega t, \sin\omega t), \text{ where } l = \frac{2\sigma_1}{\pi} = \frac{2E}{\pi T_0}, \;\; \omega = \frac{\pi c}{\sigma_1} = \frac{\pi c T_0}{E}, \tag{4.35}$$

where l is the string length and ω the angular frequency. This we find that (with $u = ct$)

$$(F^1(u), F^2(u)) = \frac{l}{2}(\cos\frac{\omega u}{c}, \sin\frac{\omega u}{c}) \;\; \Rightarrow \;\; (F'^i(u))^2 = (\frac{l\omega}{2c})^2 = 1. \tag{4.36}$$

Thus we see that the constraints are indeed satisfied. Note also that the σ dependence that appears in X^i above, that is $\cos\frac{\pi\sigma}{\sigma_1}$, is such that it parametrises the string in a 1-1 manner. Note also that by combining the different cos and sin factors in X^i one obtains exactly the function F^i with arguments $ct \pm \sigma$ and that for this particular motion of the string $v_0^i = 0$.

For the closed string we can use the solution of the wave equation given above which we now write as (using (u, v) instead of (σ^+, σ^-))

$$X^i(t,\sigma) = \frac{1}{2}(F^i(u) + G^i(v)), \quad u = ct + \sigma, \;\; v = ct - \sigma. \tag{4.37}$$

Then

$$\frac{1}{c}\partial_t X^i = \frac{1}{2}(F'^i(u) + G'^i(v)), \tag{4.38}$$

$$\partial_\sigma X^i = \frac{1}{2}(F'^i(u) - G'^i(v)), \tag{4.39}$$

The combined constraints formula above then implies

$$(F'^i)^2 = (G'^i)^2 = 1. \tag{4.40}$$

Also here we have the relation $\sigma_1 = E/T_0$.

The new feature compared to the open string is of course the (periodic) boundary condition which now reads

$$X^i(t, \sigma + \sigma_1) = X^i(t, \sigma) \;\Rightarrow\; F^i(u + \sigma_1) + G^i(v - \sigma_1) = F^i(u) + G^i(v). \tag{4.41}$$

Taking u and v derivatives we see that this implies

$$F'^i(u + \sigma_1) = F'^i(u), \;\; G'^i(v - \sigma_1) = G'^i(v), \tag{4.42}$$

so we conclude that in general the two independent sets of functions F^i and G^i are both σ_1 quasi-periodic (not $2\sigma_1$ as in the open case).

The conclusion we have arrived at is that $(F'^i)^2 = (G'^i)^2 = 1$. So if the motion takes place in a 3-dimensional space then the vectors $F'^i(u)$ and $G'^i(v)$ are both unit vectors on the sphere S^2 in $\mathbf{R}^3$. Since they are also σ_1 periodic they trace out two closed loops on S^2. No points on the closed string is forced to move with the speed of light c.

An interesting situation arises, however, if the two closed curves happen to intersect (twice). At the point(s) of intersection we have then

$$F'^i(u_0) = G'^i(v_0), \;\text{ for some values }\; (u_0, v_0), \;\text{ that is some }\; (\tau_0, \sigma_0). \tag{4.43}$$

At these points this implies

$$\partial_\sigma X^i(\tau_0, \sigma_0) = 0 \;\;\Rightarrow\;\; \text{cusp at }\; (\tau_0, \sigma_0). \tag{4.44}$$

To see that we get a cusp we expand $X^i(\tau, \sigma)$ in σ close to a point (τ_0, σ_0). This gives

$$X^i(\tau, \sigma) = X^i(\tau_0, \sigma_0) + \frac{1}{2}(\sigma - \sigma_0)^2 \partial_\sigma^2 X^i(\tau_0, \sigma_0) + ... \tag{4.45}$$

This implies directly that there is a cusp since $\partial_\sigma X^i(\tau, \sigma) = (\sigma - \sigma_0)\partial_\sigma^2 X^i(\tau_0, \sigma_0) + ...$ which is zero at σ_0 but close to it the tangent vectors at points $\sigma < \sigma_0$ and $\sigma > \sigma_0$ are anti-parallell.

It is also clear that these cusps, from $\frac{1}{c}\partial_t X^i = F'^i(u_0)$, move with the speed of light c since $|F'^i(u_0)|^2 = \frac{v^2}{c^2} = 1$.

Comment: The classical strings discussed here have no natural size and could be anything from Planck size to the size of the universe. This issue will get an answer later for the fundamental string, which is normally considered to be Planck size. At this point, however, we could ask if there exist observable string-like objects in the universe, so called **cosmic strings**. In fact, at a couple of occasions in the past pictures from various observatories were believed to prove their existence. Unfortunately, after some time other interpretations appeared which were later accepted[10].

[10] See a very recent discussion in *Cosmic String Gravitational Waves Could Solve Antimatter Mystery*, Live Science, March, 2020.

4.2 BZ Chapter 8: World-sheet currents

This chapter contains just some comments about conserved currents and other phenomena that can be derived from the Lagrangian formulation and Noether's theorem applied to the **field theory on the world-sheet**. This means that we should regard $X^\mu(\tau, \sigma)$ as fields living on a 2-dimensional space-time with coordinates (τ, σ). This 2-dimensional field theory picture of string theory is of paramount importance and something we will make use of many times in rest of the course.

Comment (important!) Having emphasised that string theory should be regarded as a 2-dimensional field theory, we will see later that it must be turned into a 2-dimensional quantum field theory. A very natural question is then: String theory is supposed to generalise Einstein's theory of gravity and turn it into a *consistent quantum gravity theory in four spacetime, or maybe higher, dimensions*? So what is going on?
Answer: The magic of string theory is that the 2-dimensional quantum field theory, where $X^\mu(\tau, \sigma)$ are the quantum fields, does indeed generate, in a well-defined way, an **effective**[11] quantum field theory in 4 or higher spacetime dimensions. How this is done is unique to string theory and it is one of the goals of this course to give a hint how this can be possible. We will not be able to explain this in detail here since that would require at least one more course of a more mathematical nature.

We have, in fact, already derived the key equations in this context when we obtained the Nambu-Goto equations of motion by varying the string action $S[X^\mu]$ (using Hamilton's principle):

$$\partial_\alpha \mathcal{P}^\alpha_\mu = 0, \text{ where } \mathcal{P}^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)}, \tag{4.46}$$

and the various possible boundary conditions, Neumann and Dirichlet for the open string.

The Lagrangian used above is the 2-dimensional Lagrangian for the string coordinates $X^\mu(\tau, \sigma)$, so the action is

$$S[X^\mu] = \int d\tau d\sigma \, \mathcal{L}, \;\; \mathcal{L} = -\frac{T_0}{c}\sqrt{(\dot{X}X')^2 - \dot{X}^2 X'^2}. \tag{4.47}$$

Note that the actual string Lagrangian does not depend on the string coordinates X^μ themselves, only their derivatives. This fact implies that there is a number of symmetries present that we need to identify and analyse. This analysis will be even more interesting in the Polyakov formulation of the string.

The picture that emerges now is the following: There are two kinds of symmetries in a string theory viewed as a 2-dimensional field theory:

[11] More information about the concept of effective field theory can be found in the lectures by A Hebecker, hep-th/2008.10625. This text is, however, quite a bit more advanced than our string course.

Local symmetries: These concern the reparametrisation invariance in τ, σ that has been discussed in detail above.

Global symmetries: These concern properties of the target space, here a Minkowski spacetime of some dimension D, that is Poincaré symmetry. Infinitesimally these are given in terms of constant parameters $\epsilon^{\mu\nu}$ and ϵ^{μ} by

$$\text{Lorentz: } \delta X^{\mu} = \epsilon^{\mu}{}_{\nu} X^{\nu}, \ \text{ translations: } \delta X^{\mu} = \epsilon^{\mu}. \tag{4.48}$$

Applying Noether's theorem (recall that since the Lagrangian is invariant and not a total derivative the conserved current has only one term)

$$j^{\alpha} := \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha} X^{\mu})} \delta X^{\mu}, \ \text{where the parameters should be dropped.} \tag{4.49}$$

The above field transformations then give two kinds of conserved currents

$$\text{Translations: } j^{\alpha}_{\mu} := \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha} X^{\mu})} = \mathcal{P}^{\alpha}_{\mu}, \tag{4.50}$$

where we have noted that we just got the quantities $\mathcal{P}^{\alpha}_{\mu}$ defined previously. The Nambu-Goto equations of motion are thus equivalent to the conservation of this current.

For Lorentz transformations enumerated by two antisymmetrised indices we get the current from the definition (note the 1/2 and that we denote the current by $\mathcal{M}^{\alpha}$ instead of j^{α}))

$$\tfrac{1}{2} \epsilon^{\mu\nu} \mathcal{M}^{\alpha}_{\mu\nu} := \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha} X^{\mu})} \delta X^{\mu}. \tag{4.51}$$

Thus, dropping the parameters on both sides,

$$\text{Lorentz: } \mathcal{M}^{\alpha}_{\mu\nu} := 2 \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha} X^{[\mu})} X_{\nu]} = \mathcal{P}^{\alpha}_{\mu} X_{\nu} - \mathcal{P}^{\alpha}_{\nu} X_{\mu}. \tag{4.52}$$

From these currents we get the charges, i.e., the generators of the respective transformations, by a σ-integration over the τ-component:

$$P_{\mu} := \int \mathcal{P}^{\tau}_{\mu} d\sigma, \quad M_{\mu\nu} := \int \mathcal{M}^{\tau}_{\mu\nu} d\sigma. \tag{4.53}$$

The Lorentz generators and their algebra will play an absolutely crucial role in development of string theory below.

We end this chapter by one observation and one comment on parameters. The observation concerns the space integral used when deriving the charges from the currents. Above we used an integral over space, i.e., over the σ-direction as in $P_{\mu} := \int \mathcal{P}^{\tau}_{\mu} d\sigma$. However, this integral can be written more generally as an integral along a general curve γ as

$$P^{(\gamma)}_{\mu} := \int_{\gamma} (\mathcal{P}^{\tau}_{\mu} d\sigma - \mathcal{P}^{\sigma}_{\mu} d\tau), \tag{4.54}$$

where γ runs from one side to the other on the open string world-sheet or once around the closed string. For the open string this is true since using another such curve $\bar{\gamma}$, and constructing a closed curve Γ by adding the world-sheet edges between the points where γ and $\bar{\gamma}$ reach the edges, we find that the closed curve Γ integral vanishes since $\oint_\Gamma(\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau) := \int_{A:\partial A=\Gamma} \partial_\alpha \mathcal{P}^\alpha_\mu d\tau d\sigma = 0$ by Stokes theorem and current conservation. The same argument works for the closed string case where Γ is constructed from two different closed loops around the world-sheet connected by a line between them run through in both directions.

We end this chapter by introducing the perhaps most famous concept in string theory, the so called **Regge slope parameter:** α'. This is important for many reasons one being that it makes explicit a new, and perhaps unfamiliar, aspect possessed by the string. Let us return to the rigid string in the static gauge, rotating in the xy-plane,:

$$X^i(t,\sigma) = \frac{\sigma_1}{\pi}\cos\frac{\pi\sigma}{\sigma_1}(\cos\frac{\pi ct}{\sigma_1}, \sin\frac{\pi ct}{\sigma_1}). \tag{4.55}$$

Using $\mathcal{P}^{\tau i} = \frac{T_0}{c}\frac{\partial X^i}{\partial t}$, we get for the angular momentum generator in the z-direction (for any t so we can set $t = 0$)

$$M_z := M_{xy} = \int_0^{\sigma_1}(X_1\mathcal{P}^\tau_2 - X_2\mathcal{P}^\tau_1)d\sigma = \tfrac{\sigma_1 T_0}{\pi c}\int_0^{\sigma_1}\cos^2\frac{\pi\sigma}{\sigma_1}\,d\sigma = \frac{\sigma_1^2 T_0}{2\pi c}. \tag{4.56}$$

So, with $\sigma_1 = E/T_0$ and renaming M_{xy} as J we have

$$J = \frac{1}{2\pi T_0 c}E^2. \tag{4.57}$$

It is now standard to introduce the Regge slope parameter α' and write this equation as

$$\frac{J}{\hbar} := \alpha'\,E^2. \tag{4.58}$$

The LHS is dimensionless so $[\alpha'] = (Nm)^{-2} = L^2$ in natural units. The relation between the tension T_0 and α' is then

$$\alpha' = \frac{1}{2\pi\,T_0\,\hbar c}. \tag{4.59}$$

It is also very useful to introduce the **string length** l_s by

$$l_s := \hbar c\sqrt{\alpha'}. \tag{4.60}$$

These equations are of course most often written using natural units (i.e., with $\hbar = c = 1$).

This gives rise to a new and perhaps unfamiliar aspect of string theory: For a massive body moving around another massive body (as, e.g., in a planetary system) the relation between angular momentum J and energy E is $E \propto J^2$, the opposite to the relation for the string. The relation with α' above is called a Regge slope relation since this relation was the one found to be satisfied by heavy hadrons discovered early on in the history of elementary particle physics: When experimentalists plotted the hadron $(masses)^2$ as a function of angular momentum J (their spin) they fell on straight lines in the diagram[12].

[12] A short historical account is given in BZ page 527.

5 Lecture 5

In this lecture we return to the discussion about light-cone coordinates from the first week. Now we will show how they can be used in a very nice way to avoid the complications that we encountered in the previous lecture trying to solve the string equations explicitly. Fixing the reparametrisation invariance by means of the static gauge led to quadratic constraints $(F'^i(u))^2 = 1$ on the space components of X^μ that we could not solve (without taking square roots). This fact will lead to enormous problems when quantising the string coordinates so if there is any better way to proceed it would be most welcome. The solution is to use light-cone coordinates in spacetime.

5.1 BZ Chapter 9: Light-cone relativistic strings

The purpose of this chapter is to explain how the use of light-cone coordinates makes it possible to explicitly obtain all the independent on-shell degrees of freedom in X^μ. The basic reason something nice happens is easily seen as follows. In the previous analysis we used the static gauge to set (with $\hbar = c = 1$ from now on)

$$X^0(\tau, \sigma) = \tau. \tag{5.1}$$

After also implementing gauge fixing conditions in the σ-direction we obtained two constraints that we combined into

$$(\dot{X}^i \pm X'^i)^2 = 1. \tag{5.2}$$

This is a quadratic constraint involving all space components of X^μ.

Let us rewrite the above equations in one single covariant looking equation

$$(\dot{X}^\mu \pm X'^\mu)^2 = 0. \tag{5.3}$$

Using the static gauge condition above then gives back the σ gauge conditions.

However, if we instead use light-cone coordinates we have, splitting $\mu = (+, -, I)$,

$$(X^\mu)^2 = -2X^+X^- + X^I X^I. \tag{5.4}$$

Applying this to the covariant looking constraints above it becomes

$$-2(\dot{X}^+ \pm X'^+)(\dot{X}^- \pm X'^-) + (\dot{X}^I \pm X'^I)(\dot{X}^I \pm X'^I) = 0. \tag{5.5}$$

Then, by imposing another version of the static gauge, namely on X^+, (for details see below)

$$X^+(\tau, \sigma) = \beta\alpha' p^+ \tau, \ \Rightarrow \ \dot{X}^+ = \beta\alpha' p^+, \ \ X'^+ = 0, \tag{5.6}$$

the quadratic constraint reads instead

$$-2\beta\alpha' p^+(\dot{X}^- \pm X'^-) + (\dot{X}^I \pm X'^I)(\dot{X}^I \pm X'^I) = 0. \tag{5.7}$$

The form used for X^+ is chosen for the following reasons:
1. From now on we will consider (τ, σ) to be dimensionless numbers.
2. Since X^+ is a component in the + direction this has to be true also for the RHS. This requires us to put in p^+ which is a component of the conserved spacetime momentum and hence a constant. We will assume $p^+ > 0$.
3. $\beta = 2$ for the open string and $\beta = 1$ for the closed string. This will be explained later.
4. With the above rules we need a factor α' for dimensional reasons.

The goal now is to use the above **light-cone gauge** to show that all (almost) independent degrees of freedom in string theory come from the **transverse** string components X^I $(I = 2, 3, ..., d)$ where we have split $\mu = (+, -, I)$. Clearly, we can now solve the quadratic constraint without any square roots appearing:

$$\dot{X}^- \pm X'^- = \frac{1}{2\beta\alpha' p^+}(\dot{X}^I \pm X'^I)(\dot{X}^I \pm X'^I). \tag{5.8}$$

This is one of the most important equations in this course!

Comment: The equations in this discussion in BZ are often written in terms of a fixed vector n^μ, for instance $X^+ := n \cdot X$ where $n^\mu \propto (1, 1, 0, ..., 0)$. One can then let this vector be time-like instead which shows that both gauge choices can be seen as just special cases of the vector n^μ. However, this does not really provide any deeper understanding so we will skip this aspect of the discussion in these notes.

Note that we cheated a bit above since we took the constraints directly from the previous analysis based on the static gauge in the time direction, $X^0 = \tau$. The arguments in the light-cone case leading to the quadratic constraints are slightly different so let us go through them briefly here. To do this we return to the original form of the theory before any gauge conditions are imposed and start over with the somewhat different steps 1, 2, and 3 (and 4 in the closed string case).

Step 1 is, as already discussed above, the static gauge in the light-cone +-direction:

$$\tau\text{-gauge: } X^+(\tau, \sigma) = \beta\alpha' p^+ \tau. \tag{5.9}$$

Then consider

$$\mathcal{P}^\tau_\mu(\tau, \sigma) = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X} \cdot X')X'_\mu - X'^2\dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}}. \tag{5.10}$$

The theory is still σ-reparametrisation invariant on the world-sheet so, at a fixed τ, we get for a change in $\sigma \to \sigma'(\sigma)$

$$\mathcal{P}^\tau_\mu(\tau, \sigma') = \frac{d\sigma}{d\sigma'}\mathcal{P}^\tau_\mu(\tau, \sigma). \tag{5.11}$$

This is a consequence of how the σ derivatives appear in the expression for $\mathcal{P}^\tau_\mu(\tau, \sigma)$.

It then follows that the function $\sigma'(\sigma)$ can be chosen such that **one** component of $\mathcal{P}^{\tau}_{\mu}(\tau,\sigma)$ can be made σ independent. We now choose $\mathcal{P}^{\tau+}$ to have this property which defines the second step: $\mathcal{P}^{\tau+} = a(\tau)$ where $a(\tau)$ is an arbitrary function.

But the function $a(\tau)$ is easily determined by integrating $\mathcal{P}^{\tau+}(\tau,\sigma)$ over the string:

$$\int_0^{\sigma_1} d\sigma\, \mathcal{P}^{\tau+}(\tau,\sigma) := p^+ = \sigma_1\, a(\tau), \;\; \Rightarrow \;\; a(\tau) = \frac{p^+}{\sigma_1}. \tag{5.12}$$

Note that p^+ is the center of mass momentum which is constant for a free string. We now also see that $\mathcal{P}^{\tau+}(\tau,\sigma)$ is independent of both τ and σ.

Step 2 is thus the choice of σ-coordinate that gives:

$$\sigma\text{−gauge: } \mathcal{P}^{\tau+}(\tau,\sigma) = \frac{\beta}{2\pi}p^+, \;\; (\beta = 1 \text{ for the closed string and } \beta = 2 \text{ for the open string}). \tag{5.13}$$

When writing this gauge condition in terms of the parameter β we have introduced the standard values of σ_1: for the open string $\sigma_1 = \pi$ and for the closed string $\sigma_1 = 2\pi$. These values are completely arbitrary but will turn out very convenient later when we start expanding X^μ in Fourier modes.

It is now possible to make use of the above gauge conditions to derive a new condition. Consider the Nambu-Goto equation for the +-component

$$\partial_\tau \mathcal{P}^{\tau+} + \partial_\sigma \mathcal{P}^{\sigma+} = 0. \tag{5.14}$$

The σ gauge condition implies that $\partial_\tau \mathcal{P}^{\tau+} = 0$ which gives

$$\partial_\sigma \mathcal{P}^{\sigma+}(\tau,\sigma) = 0. \tag{5.15}$$

So if $\mathcal{P}^{\sigma+}(\tau,\sigma) = 0$ anywhere (i.e., for any σ) on the closed or open string it is true for all (τ,σ). To see that this is the case, consider first the open string. Recall that we must impose free bc in the time-direction since a Dirichlet bc is impossible ("time cannot stop"). In the light-cone case this fact will be applied to the +-component instead. (Note that this condition involves two directions, $\mu = 0$ and $\mu = 1$, which leads to problems in some situations involving D-branes.) Now consider the equation $\dot{p}^+ := \int_0^\pi d\sigma \partial_\tau \mathcal{P}^{\tau+} = -\mathcal{P}^{\sigma+}|_0^{\sigma_1} = 0$. Clearly the conservation of p^+ is equivalent to the vanishing of $\mathcal{P}^{\sigma+}$ at the boundaries of the open string.

For the closed string the argument that we can find a point, which we will set to $\sigma = 0$, where $\mathcal{P}^{\sigma+}(\tau,\sigma) = 0$ is a bit different. Consider here the expression for $\mathcal{P}^{\sigma+}(\tau,\sigma)$, using that $\partial_\sigma X^+ = 0$,:

$$\mathcal{P}^{\sigma+}(\tau,\sigma) = -\frac{1}{2\pi\alpha'}\frac{(\dot{X}\cdot X')\partial_\tau X^+}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}. \tag{5.16}$$

So if we can show that $\dot{X} \cdot X' = 0$ at some point on the world-sheet, e.g. at $\sigma = 0$, we are home! However, this is clearly possible since we can choose two circles around the closed string world-sheet at τ and $\tau + \delta\tau$ and rotate the latter one so that its $\sigma = 0$ point is orthogonally after the $\sigma = 0$ point of the first circle. Generalising this to every other circle we find the τ line defined this way is orthogonal to the all circles at their $\sigma = 0$ points. Hence at $\sigma = 0$ (for all τ) we have $\dot{X} \cdot X'|_{\sigma=0} = 0$.

Using these results in the definition

$$\mathcal{P}^{\sigma}_{\mu} := \frac{\partial \mathcal{L}}{\partial X'^{\mu}} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\dot{X}_{\mu} - \dot{X}^2 X'_{\mu}}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}}, \tag{5.17}$$

we see that its $+$-component $\mathcal{P}^{\sigma +}$ vanishes at $\sigma = 0$. We have therefore shown that for both the open and closed strings

$$\mathcal{P}^{\sigma +}(\tau, \sigma) = 0, \text{ for all values of } \quad (\tau, \sigma). \tag{5.18}$$

Note that the above expression for $\mathcal{P}^{\sigma +}$ implies that $\mathcal{P}^{\sigma +} = 0$ is equivalent to the orthogonality condition $\dot{X} \cdot X'|_{\sigma=0} = 0$ (since in the static gauge $X'^{+} = 0$), both valid at all points on the closed string as well as the open string world-sheets. Thus
Step 3:

$$\text{Orthogonality condition}: \quad \mathcal{P}^{\sigma +}(\tau, \sigma) = \dot{X} \cdot X' = 0, \text{ for all values of } \quad (\tau, \sigma). \tag{5.19}$$

Comment: We mentioned above that there is a fourth gauge condition that needs to be discussed for the closed string. This comes from the fact that the position on the string where $\sigma = 0$ is still not chosen and thus remains a gauge freedom. However, this freedom cannot be eliminated by a gauge choice and will therefore be kept as a remaining freedom. This is a rather deep issue which will reappear later in the form of a constraint on the physical Hilbert space of states of the closed string. As such it is of course of enormous importance for the interpretation of the theory in our ordinary spacetime.

To summarise the above discussion:
1. τ-gauge
2. σ-gauge
3. Orthogonality constraint
4. Closed string $\sigma = 0$: Constraint on Hilbert space (later!)

In the light-cone context is possible to make use of the above constraints to discover that there is a further constraint. Using the orthogonality constraint $\dot{X} \cdot X' = 0$ gives

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \frac{X'^2 \dot{X}^{\mu}}{\sqrt{-\dot{X}^2 X'^2}}. \tag{5.20}$$

But the + component of this momentum is constant due to the σ gauge condition above. We thus have

$$\mathcal{P}^{\tau+} = \frac{1}{2\pi\alpha'}\frac{X'^2\dot{X}^+}{\sqrt{-\dot{X}^2X'^2}} = \frac{\beta}{2\pi}p^+. \tag{5.21}$$

Then using the τ gauge condition $X^+(\tau,\sigma) = \beta\alpha' p^+\tau$, that is $\dot{X}^+ = \beta\alpha' p^+$, the above equation becomes

$$\frac{\beta\alpha' p^+}{2\pi\alpha'}\frac{X'^2}{\sqrt{-\dot{X}^2X'^2}} = \frac{\beta p^+}{2\pi} \;\Rightarrow\; 1 = \frac{X'^2}{\sqrt{-\dot{X}^2X'^2}}. \tag{5.22}$$

We see now why we in step 1 chose the constant β to appear in the gauge choice. The last formula above can be nicely written as follows

$$\dot{X}^2 + X'^2 = 0. \tag{5.23}$$

If combined with the orthogonality constraint we therefore have following very useful form of the constraints

$$(\dot{X} \pm X')^2 = 0, \tag{5.24}$$

where the square is Lorentz invariant. This is the equation we started the whole light-cone discussion from in the beginning of this lecture!

The square root expression that appears in many formulas simplifies nicely:

$$\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2} = \sqrt{-\dot{X}^2X'^2} = \sqrt{(X'^2)^2} = X'^2. \tag{5.25}$$

This implies (note the position of the world-sheet indices)

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu, \;\; \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X'^\mu. \tag{5.26}$$

We then also find that the Nambu-Goto equation $\partial_\tau\mathcal{P}^\tau_\mu + \partial_\sigma\mathcal{P}^\sigma_\mu = 0$ reduces to the wave equation

$$\ddot{X}^\mu - X''^\mu = 0. \tag{5.27}$$

Note that in this derivation of the constraints and the wave equation they all look Lorentz covariant since they involve all components of X^μ and $\mathcal{P}^{\tau\mu}$. This is of course not entirely true since we also have the constraints on the + component of some Lorentz vectors:

$$X^+(\tau,\sigma) = \beta\alpha' p^+\tau, \;\; \mathcal{P}^{\tau+} = \frac{\beta}{2\pi}p^+, \;\; \mathcal{P}^{\sigma+} = 0. \tag{5.28}$$

Comment: The wave equation obtained in the light-cone gauge analysis above can be derived from a simple 2-dimensional Lorentz covariant Lagrangian for $X^\mu(\tau,\sigma)$

$$\mathcal{L}_{free} = -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}, \tag{5.29}$$

where we have written the two relevant flat Lorentzian metric tensors explicitly: the world-sheet $\eta_{\alpha\beta}$ and the target spacetime $\eta_{\mu\nu}$. There are several important remarks to make here:
1. We have chosen an overall minus-sign to get the space components of X^μ to have the correct sign for the Lagrangian $\mathcal{L}_{free}$. But X^0 then has an incorrect sign which ruins unitarity. This Lagrangian does therefore not define a consistent theory by itself.
2. Recall that the constraints originate from the reparametrisation invariance which is not present in the Lagrangian $\mathcal{L}_{free}$. Adding the constraints to the theory based on $\mathcal{L}_{free}$ changes it completely and turns it into a well defined 2-dimensional field theory, namely string theory. One aspect of this is that after having solved the constraints (almost) only X^I remain as the physical dof and the X^0 unitarity problem is eliminated.

Mode expansions: The extreme importance of the last comments above will become clear when we now start to solve the theory in full generality. The mode expansions obtained are among the most useful and important formulas in the whole course and perhaps in the whole subject of string theory. This will be obvious later in the course when they are used to define the world-sheet quantum field theory: Two consequences are that strings live in a the spacetime with ten dimensions and that string theory contains Einstein's theory of gravity (general relativity (GR)) as a low energy approximation. The last statement should even be understood in a much more profound way: String theory contains a generalisation of Einstein's theory (GR) that is completely consistent with quantum mechanics, i.e., it is a theory of quantum gravity.

The explicit form of the mode expansions depend on the boundary conditions for the open string and on the periodicity conditions for the closed string. We will discuss most of the possible cases in due time but start with the open string with Neumann conditions on both ends. However, in all cases it is convenient to start by solving the wave equation for **all** components of X^μ, which is an important aspect here.

As usual we do this by introducing light-cone coordinates also on the world-sheet:

$$\sigma^{\pm} = (u, v) = (\tau + \sigma, \tau - \sigma) \;\Rightarrow\; \partial_{\pm} = (\partial_u, \partial_v) := \frac{1}{2}(\partial_\tau \pm \partial_\sigma). \tag{5.30}$$

Thus the wave equation

$$\Box_2 X^\mu = -4\partial_+\partial_- X^\mu = 0 \;\Rightarrow\; X^\mu(\tau,\sigma) = \frac{1}{2}(f^\mu(u) + g^\mu(v)), \tag{5.31}$$

where $f^\mu(u)$ and $g^\mu(v)$ are arbitrary functions.

It is now fairly easy to implement any kind of boundary conditions. For Neumann bc at the $\sigma = 0$ end of the string we see that

$$\partial_\sigma X^\mu(\tau,\sigma)|_{\sigma=0} = 0 \;\Rightarrow\; f'(\tau) = g'(\tau) \text{ that is } f(\tau) = g(\tau) + c, \tag{5.32}$$

where c is a constant that can be redefined away by absorbing it into the function $f(\tau)$. Thus

$$\partial_\sigma X^\mu(\tau,\sigma)|_{\sigma=0} = 0 \;\Rightarrow\; X^\mu(\tau,\sigma) = \frac{1}{2}(f^\mu(u) + f^\mu(v)). \tag{5.33}$$

Imposing Neumann bc also at the $\sigma = \pi$ end we get

$$\partial_\sigma X^\mu(\tau,\sigma)|_{\sigma=\pi} = 0 \;\; \Rightarrow \;\; f'^\mu(\tau+\pi) - f'^\mu(\tau-\pi) = 0. \tag{5.34}$$

So we find that the functions $f'^\mu(u)$ are all 2π periodic: With $u := \tau + \sigma$ and $\sigma = \pi$ we get

$$f'^\mu(u) = f'^\mu(u - 2\pi). \tag{5.35}$$

Thus the mode expansion for the functions f'^μ read, with $n \in \mathbf{Z}^+$,

$$f'^\mu(u) = f_1^\mu + \Sigma_{n=1}^\infty (a_n^\mu \cos nu + b_n^\mu \sin nu). \tag{5.36}$$

This integrates to the following general form

$$f^\mu(u) = f_0^\mu + f_1^\mu\, u + \Sigma_{n=1}^\infty (A_n^\mu \cos nu + B_n^\mu \sin nu). \tag{5.37}$$

From this result we see that the mode expansion for the *real functions* X^μ can be written

$$X^\mu(\tau,\sigma) := x_0^\mu + 2\alpha' p^\mu \tau - i\sqrt{2\alpha'}\Sigma_{n=1}^\infty (a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau})\frac{1}{\sqrt{n}}\cos n\sigma. \tag{5.38}$$

Here we made several useful choices of the coefficients that will be explained below.

So finally, defining $\alpha_0^\mu := \sqrt{2\alpha'}p^\mu$, $a_n := \frac{1}{\sqrt{n}}\alpha_n$ and $a_n^* := \frac{1}{\sqrt{n}}\alpha_{-n}$ for $n > 0$, this expansion reads

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'}\Sigma_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma. \tag{5.39}$$

At this point it might be worth while to develop some intuition for the this expansion. The following aspects may be useful to keep in mind when the expansions are derived using other boundary conditions or for the closed string:
1. There are two fundamentally different kinds of modes in X^μ: The zero modes and the oscillator modes.
The zero modes are $x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau$ which are related to the motion of the whole string, i.e., to the center of mass. Integrating over the $X^\mu(\tau,\sigma)$ from $\sigma = 0$ to $\sigma = \pi$ eliminates all oscillator terms and thus gives

$$\int_0^\pi d\sigma X^\mu(\tau,\sigma) = \pi x_0^\mu + 2\pi\alpha' p^\mu \tau. \tag{5.40}$$

The center of mass momentum is then

$$p^\mu := \int_0^\pi d\sigma \mathcal{P}^{\tau\mu} = \int_0^\pi d\sigma \frac{1}{2\pi\alpha'}\dot{X}^\mu = p^\mu, \tag{5.41}$$

which explains the choice of coefficients multiplying p^μ in $X^\mu(\tau,\sigma)$.
2. The oscillator terms get their σ dependence directly from the boundary conditions. The oscillators a_n^μ are dimensionless which explains the α' factor. The funny factors of $\frac{1}{\sqrt{n}}$ are introduced in order to get the quantised oscillators to commute to one: $[a_n^\mu, a_m^\nu] = \eta^{\mu\nu}\delta_{n+m,0}$

which is standard in any QFT. This fact will be demonstrated later. However, in string theory it is more convenient and hence standard practise to rescale the oscillators in another way, namely by defining

$$\text{for } n \geq 1: \quad \alpha_n^\mu := \sqrt{n} a_n^\mu, \ \ \alpha_{-n}^\mu := \sqrt{n} a_n^{*\mu}. \tag{5.42}$$

This is nice since it then becomes possible to write all terms under one summation as done above.

The choice of expansion coefficients above become even more understandable if we compute

$$\dot{X}^\mu \pm X'^\mu = \sqrt{2\alpha'}\, \Sigma_{n\in\mathbf{Z}}\, \alpha_n^\mu e^{-in(\tau\pm\sigma)}, \tag{5.43}$$

where we have also defined a α_0^μ

$$\alpha_0^\mu := \sqrt{2\alpha'} p^\mu. \tag{5.44}$$

It is now time to show explicitly that the above light-cone formalism makes it possible to find all the independent degrees of freedom in the string coordinates $X^\mu(\tau,\sigma)$ and express everything else in terms of them. The end result of this analysis will give us the so called **transverse Virasoro generators**. They will be defined below and play a key role in the following lectures.

Comment: A somewhat problematic aspect of the light-cone formalism is that the results obtained are not automatically Lorentz covariant. There is a much more powerful formulation of string theory, the so called **Polyakov formulation**, which is Lorentz covariant but also more technically and mathematically demanding. We will nevertheless introduce it briefly later and try to draw some important conclusions from it. In the formulation of string theory used here, which is not **manifestly** Lorenz covariant, we must instead perform a calculation to prove Lorentz covariance.

Finally, we come to the whole point of the light-cone formulation. Consider again the constraints, splitting $\mu = (+,-,I)$ where the index I runs over the transverse directions, i.e., $I = 2, 3, ..., d$,

$$(\dot{X}^\mu \pm X'^\mu)^2 = 0 \ \Leftrightarrow\ -2(\dot{X}^+ \pm X'^+)(\dot{X}^- \pm X'^-) + (\dot{X}^I \pm X'^I)^2 = 0. \tag{5.45}$$

Then using the constraints form the gauge choices we have in general

$$\dot{X}^+ \pm X'^+ = \beta\alpha'\, p^+ > 0, \tag{5.46}$$

where p^+ is constant (in the free theory) and $p^+ > 0$ (which should be regarded as generally true). Thus we find

$$\dot{X}^- \pm X'^- = \frac{1}{2\beta\alpha'\, p^+}(\dot{X}^I \pm X'^I)^2. \tag{5.47}$$

Since we have now solved the wave equations and all the constraints on X^μ we can identify the independent degrees of freedom:

$$X^I(\tau, \sigma), \;\; p^+, \;\; x_0^-, \tag{5.48}$$

where all modes in X^I are independent of each other and of p^+, x_0^-. It is very important that we are able to group all these into canonical pairs like (q, p) and $(a, a^\dagger)$. But this is easily done (compare to QFT in Minkowski space):

$$(x_0^-, p^+), \;\; (x_0^I, p^I), \;\; (\alpha_n^I, \alpha_{-n}^I), \;\; n = 1, 2, 3... \tag{5.49}$$

Note that this indicates that we will regard α_{-n}^I for $n \geq 1$ as creation operators and α_n^I for $n \geq 1$ as annihilation operators.

Since we have obtained the mode expansion for the open string with (N, N) bc above we specialise to the open string by setting $\beta = 2$ and insert the above expansions for X^I into the expression for $\dot{X}^- \pm X'^-$ given above. We get

$$\dot{X}^- \pm X'^- = \frac{1}{4\alpha' p^+}(\dot{X}^I \pm X'^I)^2 := \frac{1}{p^+}\Sigma_{n\in\mathbf{Z}}\, L_n^\perp e^{-in(\tau\pm\sigma)}, \tag{5.50}$$

where we have defined the **transverse Virasoro generators** in the light-cone gauge (indicated by the $\perp$) by

$$L_n^\perp := \frac{1}{2}\Sigma_{p\in\mathbf{Z}}\alpha_{n-p}^I \alpha_p^I. \tag{5.51}$$

Exercise: Show that this form of $L_n^\perp$ is correct!

Using again $\dot{X}^\mu \pm X'^\mu = \sqrt{2\alpha'}\,\Sigma_{n\in\mathbf{Z}}\,\alpha_n^\mu e^{-in(\tau\pm\sigma)}$, we find that all the oscillators in X^- are expressed in terms of the independent degrees of freedom through, for all $n \in \mathbf{Z}$,

$$\sqrt{2\alpha'}\alpha_n^- = \frac{L_n^\perp}{p^+}. \tag{5.52}$$

In particular for $n = 0$ we get (recall $\alpha_0^\mu := \sqrt{2\alpha'}p^\mu$)

$$\sqrt{2\alpha'}\alpha_0^- = \frac{L_0^\perp}{p^+} \;\; \Rightarrow 2p^+p^- = \frac{1}{\alpha'}L_0^\perp = p^I p^I + \frac{1}{\alpha'}\Sigma_{n=1}^\infty \alpha^{*I}_{\;n}\alpha_n^I. \tag{5.53}$$

This equation tells us that the spacetime mass spectrum of the string is given by

$$M^2 = -p^2 = 2p^+p^- - p^I p^I = \frac{1}{\alpha'}\Sigma_{n=1}^\infty \alpha^{*I}_{\;n}\alpha_n^I. \tag{5.54}$$

We can hence conclude that at the classical level the string only contains configurations that satisfy $M^2 \geq 0$, where the actual value can vary continuously.

It will be one of our main goals in this course to understand this result for M^2 at the quantum level. This requires a full and exact QFT treatment of the 2-dimensional world-sheet field theory based on $X^\mu(\tau,\sigma)$ and the mode expansion given above. This discussion will be started in the next lecture by first looking at the point particle and then we redo it for the string. It is here that the "famous" identity for the sum over all positive integers appears:

$$\Sigma_{n=1}^{\infty} n = -\tfrac{1}{12}. \tag{5.55}$$

5.2 BZ Chapter 10: Light-cone fields and particles

This chapter contains a lot of stuff that should be familiar to most people having some basic training in QFT. Therefore, here we will only go through the things that might be less known, namely the light-cone treatment of some field theories that will appear later when discussing the field theory spectrum of string theory.

We start by considering the simplest example which is Maxwell's theory. The Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \;\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.56}$$

The field strength is invariant under the gauge transformation

$$\delta A_\mu(x) = \partial_\mu \epsilon(x). \tag{5.57}$$

A general variation δA_μ of the Lagrangian density gives

$$\delta\mathcal{L} = -\frac{1}{2}F_{\mu\nu}\delta F^{\mu\nu} = -F^{\mu\nu}\partial_\mu \delta A_\nu = (\partial_\mu F^{\mu\nu})\delta A_\nu - \partial_\mu(F^{\mu\nu}\delta A_\nu). \tag{5.58}$$

Setting the variation of the action $\delta S[A_\mu] = \int d^4x \delta\mathcal{L} = 0$ implies the field equations, i.e., Maxwell's equations,

$$\partial_\mu F^{\mu\nu} = 0. \tag{5.59}$$

As is well-known the photon has two degrees of freedom (dof) in four spacetime dimensions. In D dimensions A_μ has instead $D-2$ dof which we now shall prove using light-cone techniques. To do this we write out Maxwell's equations explicitly in terms of A_μ

$$\Box A_\mu - \partial_\mu(\partial_\nu A^\nu) = 0. \tag{5.60}$$

After Fourier transformation to momentum space they read (we keep the symbol A_μ in momentum space)

$$k^2 A^\mu - k^\mu(k_\nu A^\nu) = 0. \tag{5.61}$$

Using light-cone coordinates, with $A^\mu = (A^+, A^-, A^I)$, these equations become

$$\mu = + : \;\; (-2k^+k^- + k^Ik^I)A^+ - k^+(-k^+A^- - k^-A^+ + k^IA^I) = 0, \tag{5.62}$$

Looking at this equation we see immediately what we should do here: If we use the gauge invariance in momentum space $\delta A^\mu(x) = -ik^\mu\epsilon(x)$ for $\mu = +$ to set $A^+ = 0$ (remember that we always assume $k^+ > 0$ so that we can divide by it). Then the above Maxwell equation for $\mu = +$ is easily solved giving

$$A^- = \frac{1}{k^+}(k^IA^I). \tag{5.63}$$

Thus we are already able to conclude that all independent dof reside in X^I which are $D-2$ in number. We have of course to check the remaining Maxwell equations so let us do that. The $\mu = -$ and $\mu = I$ components read

$$\mu = - : \;\; (-2k^+k^- + k^Ik^I)A^- - k^-(-k^+A^- - k^-A^+ + k^IA^I) = 0, \tag{5.64}$$

$$\mu = I: \quad (-2k^+k^- + k^Ik^I)A^I - k^I(-k^+A^- - k^-A^+ + k^IA^I) = 0. \tag{5.65}$$

Thus we see that everything is OK: The second bracket in these two equation is the same as the equation for the + component and thus vanishes when using the gauge condition $A^+ = 0$. Then A^I satisfy the Klein-Gordon equation which implies that this is also the case for A^-.

In the context of QFT, which means QED in this case, we should quantise all these independent degrees of freedom. This gives rise to the following state space (1-particle Hilbert space):

$$|p^+, p^I; I\rangle = a^{I\dagger}_{(p^+,p^I)}|0\rangle, \quad I = 2, 3, ..., d, \tag{5.66}$$

where the light-cone energy is $p^- = \frac{1}{2p^+}p^Ip^I$ (from $p^2A^I = 0$) and $|0\rangle$ is the perturbative QFT vacuum.

This exercise can be repeated for gravity then applying the light-cone method to the linearised Einstein's equations. From any course in GR we know that these equations arise from writing the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and expanding Einstein's equations to first order in $h_{\mu\nu}$. This gives after Fourier transforming to momentum space

$$p^2h^{\mu\nu} - (p^\mu p_\rho h^{\rho\nu} + p^\nu p_\rho h^{\rho\mu}) + p^\mu p^\nu h = 0, \tag{5.67}$$

where $h := \eta_{\mu\nu}h^{\mu\nu}$. The gauge invariance, which comes from linearised coordinate transformations, read

$$\delta h^{\mu\nu} = ip^\mu\epsilon^\nu + ip^\nu\epsilon^\mu. \tag{5.68}$$

It is easy to check that this transformation leaves the above linearised Einstein equations invariant.

The situation is now similar to the one for the Maxwell theory discussed above but since there are more indices here the analysis is a bit more involved. The first step is to check which components of $h^{\mu\nu}$ we can gauge to zero. Recalling that it is only p^+ that we can divide by we see from

$$\delta h^{++} = 2ip^+\epsilon^+, \quad \delta h^{+-} = ip^+\epsilon^- + ip^-\epsilon^+, \quad \delta h^{+I} = ip^+\epsilon^I + ip^I\epsilon^+, \tag{5.69}$$

that the parameters ϵ^+, ϵ^-, ϵ^I can be used to set

$$h^{++} = h^{+-} = h^{+I} = 0. \tag{5.70}$$

This is the full set of gauge conditions since we have used up all the parameters ϵ^μ.

Inserting these gauge conditions into the $\mu = +$ field equations

$$\mu = + \ \Rightarrow \ p^2h^{+\nu} - (p^+p_\rho h^{\rho\nu} + p^\nu p_\rho h^{\rho+}) + p^+p^\nu h = 0 \tag{5.71}$$

gives the conditions

$$p^+p_\rho h^{\rho\nu} - p^+p^\nu h = 0, \ \text{ where now } \ h = \eta_{\mu\nu}h^{\mu\nu} = h^I{}_I. \tag{5.72}$$

Thus

$$\nu = + \;\Rightarrow\; h = h^I{}_I = 0,\;\; p_\mu h^{\mu\nu} = 0. \tag{5.73}$$

This directly means that all components satisfy the Klein-Gordon equation

$$\Box h^{\mu\nu} = 0. \tag{5.74}$$

We must now check which of the non-zero tensor components of $h^{\mu\nu}$ are independent dof. However, the only restriction left on these come from $p_\mu h^{\mu\nu} = 0$, i.e.,

$$-p^+ h^{-\nu} - p^- h^{+\nu} + p^I h^{I\nu} = 0 \;\Rightarrow\; h^{-\nu} = \frac{1}{p^+} p^I h^{I\nu}. \tag{5.75}$$

Together with the gauge choices above $h^{+\nu} = 0$ we therefore conclude that the independent dof in gravity are

$$\text{Gravity independent dof: } h^{IJ},\;\; h^I{}_I = 0. \tag{5.76}$$

Note that this field transforms as an irreducible representation of the rotation group $SO(D-2)$ acting on the transverse indices. This group is called *the little group*.

The corresponding state space consists of states of the form (where tilde means tracelessness)

$$\Sigma_{I,J} \xi_{(\tilde{IJ})} a^{(\tilde{IJ})\dagger}_{(p^+,p^I)} |0\rangle. \tag{5.77}$$

The number of such states in a D-dimensional spacetime is $n = \frac{1}{2}(D-2)(D-1) - 1 = \frac{1}{2}D(D-3)$. Some relevant examples are $D = 4$ gives $n = 2$, $D = 10$ gives $n = 35$ and $D = 11$ gives $n = 44$, where the last two cases apply to string theory and M-theory, respectively. One may also note the very interesting consequences that $D = 3$ gives $n = 0$ and that n is negative for $D = 2$.

A third example of a field theory that plays a fundamental role in both string theory and M-theory is the Kalb-Ramond field $B_{\mu\nu}$. This field is special since it is an antisymmetric gauge field, i.e., $B_{\mu\nu} = -B_{\nu\mu}$ and $\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$. The light-cone analysis of this field is the subject of a home problem and will not be done here.

6 Lecture 6

In this and the next lecture we will take the crucial step of quantising the string theory. We consider first the relativistic point particle and then the string. The reason for this is that the point particle is much simpler than string theory but contains nevertheless many of the key features of quantum string theory.

6.1 Chapter 11: The relativistic quantum point particle

The action functional for the relativistic point particle moving in a Minkowski spacetime of dimension D is, as has been discussed above, given by the proper length

$$S = -m \int_{\tau_i}^{\tau_f} ds(x^\mu(\tau)) = -m \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\, d\tau. \tag{6.1}$$

Here the τ-dependence of $x^\mu(\tau)$ represents some arbitrary parametrisation of the path of the particle. The action is independent of the choice of this parameter. (Note that τ is not the proper length here although this use of τ is common in other books on the subject.) The integrand of this action is the square root of the "determinant" of the 1-dimensional metric $g_{\tau\tau} := \dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu}$ obtained as the pull-back of the target spacetime metric $\eta_{\mu\nu}$.

Thus we write, in a bit more compact notation,

$$S = \int L\, d\tau, \;\; L = -m\sqrt{-\dot{x}^2}. \tag{6.2}$$

The canonical momenta are then

$$p_\mu := \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}. \tag{6.3}$$

Comment: Viewed as a 1-dimensional field theory on the world-line there is no distinction between time and space components in the target spacetime. Thus we need the canonical momenta for all "velocities" $\dot{x}^\mu$. This is contrary to what we did previously when we expressed this action in terms of t, time in spacetime, and used only the "space" velocities $v^i := \frac{dx^i}{dt}$. After choosing t as the parameter on the world-line the theory is no longer reparametrisation invariant.

Squaring the equation for the momenta above gives the condition (called "the on-shell condition" in field theory)

$$p^2 + m^2 = 0. \tag{6.4}$$

The Lagrange equations $\partial_\tau \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0$ give directly the free equation

$$\dot{p}^\mu = 0. \tag{6.5}$$

Reparametrisation invariant theories have the very strange feature that the (naive) Hamiltonian vanishes. In the case studied here the Hamiltonian for τ-translations

$$H = p_\mu\dot{x}^\mu - L = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}\dot{x}^\mu - (-m\sqrt{-\dot{x}^2}) = 0. \tag{6.6}$$

So what happens to time evolution if this Hamiltonian vanishes? Remember that τ can always be related to t by some monotonous function.

Theories that contain conditions on the phase space variables (x^μ, p^μ), like $p^2 + m^2 = 0$ above, are called **constrained systems** and require a deeper analysis developed by Dirac. This method will not be used here but we will use one consequence of it namely that one has to introduce a new condition, a gauge condition, involving some component of x^μ, and τ.

Since we have already seen that it is useful to use the light-cone coordinates in this context we impose the *static light-cone gauge condition* (m^2 is needed since τ is dimensionless here)

$$x^+(\tau) = \frac{p^+\tau}{m^2}. \tag{6.7}$$

This gauge condition implies

$$p^+ = \frac{m\dot{x}^+}{\sqrt{-\dot{x}^2}} = \frac{p^+}{m\sqrt{-\dot{x}^2}} \;\Rightarrow\; m\sqrt{-\dot{x}^2} = 1. \tag{6.8}$$

Then this result, and the above Lagrange equation, give

$$p^\mu = m^2\dot{x}^\mu \;\Rightarrow\; \ddot{x}^\mu = 0,\;\; p^2 + m^2 = 0. \tag{6.9}$$

A full Dirac analysis[13] of this theory would produce the Hamiltonian $H = \frac{1}{2m^2}(p^Ip^I + m^2)$. Fortunately, there is another argument for this result based on the gauge condition: $x^+(\tau) = \frac{p^+}{m^2}\tau$. In terms of derivatives this can be written as (using $p^2 + m^2 = 0$)

$$\partial_\tau = \frac{p^+}{m^2}\partial_+ \;\Rightarrow\; H = \frac{p^+}{m^2}p^- = \frac{1}{2m^2}(p^Ip^I + m^2). \tag{6.10}$$

The implication follows from defining H as the τ translation generator and p^- as the translation generator for x^+ as we did in a previous lecture.

We have now found the independent degrees of freedom that we need to turn into quantum mechanical operators, namely x^I, p^I and x^-, p^+. Exactly how we choose them must however be compatible with the Hamiltonian $H = \frac{1}{2m^2}(p^Ip^I + m^2)$ found above. The non-triviality of this comes from the fact that, while $[x^I, H] = i\frac{p^I}{m^2}$, we have a vanishing result $[x^-, H] = 0$.

Consider now the operators defined by the solutions of the equation of motion $\ddot{x}^\mu = 0$, that is $x^\mu(\tau) = x_0^\mu + \frac{p^\mu}{m^2}\tau$ (where both x_0^μ and p^μ are constants). From these equations we conclude that in the transverse directions the canonical coordinates should be $x^I(\tau)$ but in the minus direction it must be x_0^-: The non-zero canonical commutation relations (CCR) must therefore be (for any τ)

$$[x^I, p^J] = i\eta^{IJ} = i\delta^{IJ},\;\; [x_0^-, p^+] = i\eta^{-+} = -i. \tag{6.11}$$

[13]This would involve constructing the so called *Dirac bracket* in terms of the ordinary Poisson bracket.

These CCRs are τ-independent in the S-picture and "equal time" in the H-picture.

Note, however, that $x^+(\tau)$ and $x^-(\tau)$ are special due to the static gauge condition $x^+(\tau) = \frac{p^+}{m^2}\tau$ (there is no x_0^+) and that $x^-(\tau)$ is τ independent since H does not contain p^+ and we get $[x^-(\tau), H] = 0$. So $x^-(\tau) = x_0^-$. p^- does not play an independent role since it is replaced by its solution of $p^2 + m^2 = 0$ in terms of p^+ and p^I.

Aspects of quantisation and gauge fixing: There are some non-trivial, and quite interesting, aspects related to quantisation of theories with gauge fixing (i.e., constrained systems) that we will now discuss. First we recall the meaning of **quantisation**: Consider the Poisson bracket in classical mechanics

$$\{A, B\}_{PB} := \frac{\partial A}{\partial x^\rho}\frac{\partial B}{\partial p_\rho} - \frac{\partial A}{\partial p_\rho}\frac{\partial B}{\partial x^\rho} \;\Rightarrow\; \{x^\mu, p_\nu\}_{PB} = \delta^\mu_\nu. \tag{6.12}$$

The transition to quantum mechanics ("quantisation") is done by the replacement

$$\{A, B\}_{PB} \;\rightarrow\; \frac{1}{i\hbar}[\hat{A}, \hat{B}] \;\Rightarrow [\hat{x}^\mu, \hat{p}_\nu] = i\hbar\delta^\mu_\nu, \tag{6.13}$$

where $\hat{A}$ etc are operators. We will, however, not use the hat notation unless it is absolutely necessary to avoid confusion. It is then immediately clear that this quantisation prescription is often ambiguous: Consider the function $f(x,p) = xp$ defined on the classical phase space. What is its quantum analogue? This issue will be important below. One aspect to keep in mind here is that the fundamental theory is QM not the classical one: The connection is $QM \Rightarrow$ *classical* and there is no implication in the other direction.

Now we return to the relativistic point particle, described by $L = -m\sqrt{\dot{x}^2}$, and note that it is not only spacetime Lorentz invariant but also invariant under the global target spacetime symmetry (translations) $\delta x^\mu = \epsilon^\mu$ where ϵ^μ are constants.

Noether's theorem then implies that $p^\mu := \frac{\partial L}{\partial \dot{x}^\mu}$ are conserved currents, i.e., $\partial_\tau p^\mu = 0$. The point we want to emphasise now is that the conserved charges (here just the currents themselves since the 1-dimensional theory on the world-line has no space direction) are also generators of the symmetries that gave rise to them via Noether's theorem.
We then conclude that the generator of the symmetry $\delta x^\mu = \epsilon^\mu$ is given simply by the $i\epsilon^\mu p_\mu$ since (setting $\hbar = 1$ as usual)

$$\delta_\epsilon x^\mu := [i\epsilon^\nu p_\nu, x^\mu] = -i\epsilon^\nu[x^\mu, p_\nu] = \epsilon^\mu. \tag{6.14}$$

This is the correct *covariant* result, i.e., when treating all of x^μ, p^μ as independent canonical operators.

The question is then: What happens in the light-cone gauge where the translation operator reads $i(-\epsilon^+ p^- - \epsilon^- p^+ + \epsilon^I p^I)$ with p^- given in terms of p^+, p^I using $p^2 + m^2 = 0$?

Clearly with $\epsilon^\mu = (0, 0, \epsilon^I)$ we get the correct result $\delta x^I = \epsilon^I$. However, both ϵ^+ and ϵ^- should give a zero result on x^I if the covariant result above is still to be valid. Let us therefore check what the effect of the ϵ^+ component is: (recall $[x^I, p^J] = i\eta^{IJ} = i\delta^{IJ}$)

$$\delta_{\epsilon^+} x^I = i[-\epsilon^+ p^-, x^I] = -i\epsilon^+[\frac{1}{2p^+}(p^J p^J + m^2), x^I] = -i\frac{\epsilon^+}{2p^+}[p^J p^J, x^I] = -\frac{\epsilon^+}{p^+}p^I. \quad (6.15)$$

A similar computation gives

$$\delta_{\epsilon^+} x_0^- = -\frac{\epsilon^+}{p^+}p^-. \quad (6.16)$$

To get the latter result we used (recall $[x_0^-, p^+] = i\eta^{-+} = -i$)

$$[x_0^-, \frac{1}{p^+}] = \frac{1}{p^+}p^+ x_0^- \frac{1}{p^+} - \frac{1}{p^+}x_0^- p^+ \frac{1}{p^+} = \frac{1}{p^+}[p^+, x_0^-]\frac{1}{p^+} = \frac{i}{(p^+)^2}. \quad (6.17)$$

Both of these are zero in the covariant calculation above since in that case it is only x^+ that is effected by an ϵ^+ translation: $\delta_{\epsilon^+} x^+ = \epsilon^+$. However, if we perform this computation in the light-cone gauge we get

$$\delta_{\epsilon^+} x^+ = [-i\epsilon^+ p^-, x^+] = [-i\epsilon^+ p^-, \frac{p^+}{m^2}\tau] = 0. \quad (6.18)$$

This result also contradicts the covariant result above. How are we to interpret these light-cone results?

The key property of the light-cone gauge formalism that explains what is going on is the dual role played by the operator p^- as seen from the *covariant formalism*:
1) p^- generates the translations $\delta_{\epsilon^+} x^+ = \epsilon^+$, but
2) p^- also generates τ reparametrisations via the Hamiltonian $H = \frac{p^+}{m^2}p^-$.
We can express this dual role of p^- by redefining the transformation as the sum of two terms

$$\delta_+ x^\mu := \delta_{\epsilon^+} x^\mu + \delta_\lambda x^\mu = \epsilon^\mu + \lambda \partial_\tau x^\mu. \quad (6.19)$$

The light-cone gauge requires us to impose $x^+ = \frac{p^+}{m^2}\tau$. Then in order to keep this gauge condition satisfied when doing p^- transformations we must also set

$$\delta_+ x^+ = 0 \;\Rightarrow\; \epsilon^+ + \lambda\partial_\tau x^+ = \epsilon^+ + \lambda\frac{p^+}{m^2} = 0 \;\Rightarrow\; \lambda = -\frac{m^2\epsilon^+}{p^+}. \quad (6.20)$$

The λ term is called a **compensating** reparametrisation for this reason. With this new definition the transformations generated by $i\epsilon^\mu p_\mu$ must be complemented by a compensating reparametrisation in τ with the above parameter λ. In particular we see that the ϵ^+ transformations we started this discussion with above now give the correct result.

Lorentz generators: The previous discussion about the role of compensating reparametrisation shows that gauge theories are rather complicated theories. Another aspect of this arises when trying to define the quantum version of the generators of Lorentz transformations. This discussion is however crucial for the whole interpretation and understanding of both the point particle and the string so this must be done very carefully.

Recall the definition of the covariant Lorentz generators (using $[x^\mu, p_\nu] = i\delta^\mu_\nu$)

$$M^{\mu\nu} := x^\mu p^\nu - x^\nu p^\mu \ \Rightarrow [M^{\mu\nu}, x^\rho] = i\eta^{\mu\rho}x^\nu - i\eta^{\nu\rho}x^\mu. \tag{6.21}$$

Note that these operators have no ordering ambiguities if quantised covariantly. This implies that the Lorentz generators $-\frac{i}{2}\epsilon^{\mu\nu}M_{\mu\nu}$ give

$$\delta_L x^\rho := [-\tfrac{i}{2}\epsilon^{\mu\nu}M_{\mu\nu}, x^\rho] = \epsilon^\rho{}_\nu x^\nu. \tag{6.22}$$

The Lorentz algebra is easily derived using these generators and reads

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma} + \text{three terms needed for antisymmetry}. \tag{6.23}$$

Since the Lorentz transformations represent physical symmetries they must be true also after choosing a gauge. Is this true in the light-cone gauge? In light-cone coordinates the different generators are

$$M^{IJ},\ M^{+I},\ M^{-I},\ M^{+-}. \tag{6.24}$$

So, we must show that they really do generate, for instance,

$$[M^{+-}, M^{+I}] = iM^{+I},\ \ [M^{-I}, M^{-J}] = 0. \tag{6.25}$$

The last commutator is trivially zero in the covariant formalism but not so in the light-cone formalism. There are in fact at least three issues that we need to sort out before we can start checking that $[M^{-I}, M^{-J}] = 0$:
1) How are $M^{\mu\nu}$ defined in the light-cone gauge?
2) What transformations on x^μ do they generate in the light-cone gauge?
3) Do they generate the correct Lorentz algebra in the light-cone gauge?

The answer is that we must require that:
3) $M^{\mu\nu}$ must give the whole Lorentz algebra for physical reasons, as mentioned above.
2) Lorentz transformation on x^μ which, however, should include compensating τ-reparametrisations to "stay in the gauge".
1) ???? (See below!)

Consider first M^{+-} to see what the problem is. Explicitly this generator reads

$$M^{+-}(\tau) = x^+(\tau)p^-(\tau) - x^-(\tau)p^+(\tau) = \frac{p^+}{m^2}\tau p^- - (x_0^- + \frac{p^-}{m^2}\tau)p^+ = -x_0^- p^+. \tag{6.26}$$

This looks like a nice result but it is not hermitian (not a problem in covariant quantisation)!

But this is easily fixed: We simply define a new M^{+-} with the same classical limit:

$$M^{+-} := -\frac{1}{2}(x_0^- p^+ + p^+ x_0^-). \tag{6.27}$$

The point here is that the standard *quantisation procedure*, i.e., turning Poisson brackets into quantum commutators, have ordering ambiguities (as we see here) and the hermiticity requirement is needed to lift this ambiguity.

In a similar way we define

$$M^{-I} := x_0^- p^I - \frac{1}{2}(x^I p^- + p^- x^I), \tag{6.28}$$

which is needed since $p^- := \frac{1}{2p^+}(p^I p^I + m^2)$ in the light-cone gauge.

The commutator between two Lorentz generators of the last kind must vanish as we saw above. This can indeed be shown to be true after some algebra (home problem). In string theory this calculation is an entirely different story and leads to some remarkable consequences:
1) Spacetime must have dimension $D = 26$, called the **critical dimension**, and
2) the mass spectrum, i.e., M^2 which is discrete after quantisation, is shifted in a way that implies that the states with two symmetrised vector indices in the closed string become massless. These states correspond to the graviton and prove that string theory, without assuming it, contains the metric and hence general relativity. As will be discussed more later (if time permits) Einstein's theory of gravity is, however, just a low-energy approximation of the full gravity theory contained in string theory.

Both of these two results in string theory will be discussed, but not fully derived, in the next chapter of BZ.

7 Lectures 7

7.1 Chapter 12: The relativistic quantum open string

Having analysed the quantum point particle we now turn to the first really stringy subject, the quantum relativistic string. The quantisation of string theory is the step that is absolutely necessary to get any understanding at all of its deep physical content, as for instance the fact that it contains general relativity in a way that is consistent with quantum gravity in 4 spacetime dimensions.

We have in fact already done a lot of the required work when we analysed the string in the light-cone gauge. Let us recall what the end-result of this was (we concentrate on the open string here so we set $\beta = 2$): The light-cone gauges in the τ and σ directions imply

$$X^+(\tau,\sigma) = 2\alpha' p^+ \tau, \;\; \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu, \;\; \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X'^\mu. \tag{7.1}$$

Using these equations we then found that the Nambu-Goto equations simplified to the wave equations and that the constraints became

$$\ddot{X}^\mu - X''^\mu = 0, \;\; (\dot{X}^\mu \pm X'^\mu)^2 = 0. \tag{7.2}$$

We emphasise again that all important properties of string theory come from the constraints $(\dot{X}^\mu \pm X'^\mu)^2 = 0$. In fact, dropping them gives a theory that is free but inconsistent (having kinetic terms with the wrong sign). The profound implications of the constraints will become clear below.

So let us recall also the next step which is to solve the constraints in the light-cone gauge. Thus we write the constraints as follows (assuming $p^+ > 0$)

$$\dot{X}^- \pm X'^- = \frac{1}{4p^+\alpha'}(\dot{X}^I \pm X'^I)^2. \tag{7.3}$$

Adding these two constraints we get

$$\dot{X}^- = \frac{1}{4p^+\alpha'}((\dot{X}^I)^2 + (X'^I)^2). \tag{7.4}$$

This is better written as

$$\mathcal{P}^{\tau-} = \frac{\pi}{2p^+}\left((\mathcal{P}^{\tau I})^2 + \frac{(X'^I)^2}{(2\pi\alpha')^2}\right), \tag{7.5}$$

since now the canonical variables are made explicit: the canonical momenta are $\mathcal{P}^{\tau I}$ while X'^I are just a σ derivative of the canonical coordinates.

From the above equations we can extract the independent degrees of freedom in the string:

$$X^I(\sigma), \;\; \mathcal{P}^{\tau I}(\sigma), \;\; x_0^-, \;\; p^+. \tag{7.6}$$

This conclusion follows since $X^+(\tau,\sigma)$ is completely determined by the light-cone gauge and thus given by p^+. The remaining $X^-(\tau,\sigma)$ is much more complicated but from the above equation for $\dot{X}^-$ it follows that all modes except x_0^- are determined by the independent dof in $X^I(\sigma)$, $\mathcal{P}^{\tau I}(\sigma)$ and p^+. It is nice to find that all dof fit into canonical pairs which they must.

We are now ready to quantise the theory, i.e., turning all the dof above into quantum operators satisfying standard canonical commutation relation (CCR). Thus we have (with no hats on operators) the equal "time" (that is τ which is suppressed) CCR

$$[x_0^-,p^+]=-i, \quad [X^I(\sigma),\mathcal{P}^{\tau J}(\sigma')]=i\delta^{IJ}\delta(\sigma-\sigma'), \tag{7.7}$$

with all other commutators vanishing.

Below we will discuss the mode expansion in detail but first we make a few observations above the theory given by the equations above. Our first observation is about the Hamiltonian. Recall that the light-cone gauge implies

$$X^+(\tau,\sigma)=2\alpha' p^+\tau \;\Rightarrow\; \partial_\tau=2\alpha' p^+\partial_+ \;\Rightarrow\; H=2\alpha' p^+p^-. \tag{7.8}$$

Using the expression for p^- above we find that

$$H=2\alpha' p^+\int_0^\pi d\sigma\,\mathcal{P}^{\tau -}=\pi\alpha'\int_0^\pi\left(\mathcal{P}^{\tau I}(\sigma)\mathcal{P}^{\tau I}(\sigma)+\frac{X'^I(\sigma)X'^I(\sigma)}{(2\pi\alpha')^2}\right), \tag{7.9}$$

where we have suppressed the τ dependence in the integrand since H is independent of τ. This Hamiltonian, and the canonical momenta above, follow directly from the action

$$S[X^I]=\frac{1}{4\pi\alpha'}\int d\tau\int_0^\pi d\sigma(\dot{X}^I\dot{X}^I-X'^IX'^I). \tag{7.10}$$

If we now use the mode expansion derived previously for (N,N) bc previously

$$(N,N):\quad X^\mu(\tau,\sigma)=x_0^\mu+2\alpha' p^\mu\tau+i\sqrt{2\alpha'}\Sigma_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma, \tag{7.11}$$

we can rather easily get a mode expansion also for H. Let us first express the mode expansion in exponentials and $\tau\pm\sigma$:

$$X^\mu(\tau,\sigma)=x_0^\mu+\alpha' p^\mu((\tau-\sigma)+(\tau+\sigma))+i\sqrt{\frac{\alpha'}{2}}\Sigma_{n\neq 0}\frac{1}{n}\alpha_n^\mu(e^{-in(\tau-\sigma)}+e^{-in(\tau+\sigma)}). \tag{7.12}$$

Then it is immediately clear that

$$(\partial_\tau\pm\partial_\sigma)X^\mu(\tau,\sigma)=2\alpha' p^\mu+\sqrt{2\alpha'}\Sigma_{n\neq 0}\alpha_n^\mu e^{-in(\tau\pm\sigma)}. \tag{7.13}$$

Thus if we define also $\alpha_0^\mu:=\sqrt{2\alpha'}p^\mu$ these expansions read

$$(\partial_\tau\pm\partial_\sigma)X^\mu(\tau,\sigma)=\sqrt{2\alpha'}\Sigma_{n\in Z}\alpha_n^\mu e^{-in(\tau\pm\sigma)}. \tag{7.14}$$

We want to square the $\mu = I$ components of this expansion. We find

$$((\partial_\tau \pm \partial_\sigma)X^I)^2 = 2\alpha' \Sigma_{m,p \in Z} \alpha^I_m \, \alpha^I_p e^{-im(\tau \pm \sigma)} e^{-ip(\tau \pm \sigma)}. \tag{7.15}$$

Define now the dummy summation variable $n := m + p$. Then the dubble sum above becomes

$$((\partial_\tau \pm \partial_\sigma)X^I)^2 = 2\alpha' \Sigma_{n,p \in Z} \alpha^I_{n-p} \, \alpha^I_p e^{-in(\tau \pm \sigma)} := 4\alpha' \Sigma_{n \in Z} \, L^\perp_n e^{-in(\tau \pm \sigma)}, \tag{7.16}$$

where the transverse Virasoro generators are

$$L^\perp_n := \frac{1}{2} \Sigma_{p \in Z} \, \alpha^I_{n-p} \, \alpha^I_p. \tag{7.17}$$

The open string ($\beta = 2$) solution to the constraints $\dot{X}^- \pm X'^- = \frac{1}{4p^+\alpha'}(\dot{X}^I \pm X'^I)^2$ then implies (using the same expansion for the LHS as we did for the RHS):

$$\sqrt{2\alpha'} \alpha^-_n = \frac{1}{p^+} L^\perp_n. \tag{7.18}$$

This is one of the most important equations so far.

The Hamiltonian is now nicely expressed as

$$H = 2\alpha' \, p^+ p^- = \sqrt{2\alpha'} \, p^+ \alpha^-_0 = L^\perp_0. \tag{7.19}$$

The oscillators α^I_n appearing inside $L^\perp_0$ do not all commute so, as is standard in QFT, the Hamiltonian has ordering problems. In the particular case of string theory this problem cannot be defined away as in QED in Minkowski but has to be dealt with in a well-defined manner. This leads to a lot of interesting mathematics that we will soon discuss.

Using the functional CCRs above we can also prove that the Hamiltonian implies the Klein-Gordon equation for $\mu = I$ components of X^μ. This is exactly as in ordinary QFT and will not be shown in detail here.

What we will do in detail is, however, to derive the CCRs for the oscillators α^I_n from the functional CCRs given above. This will explain some aspects of the form of the mode expansion (e.g., the $1/n$ and the range of σ for the open string) but to do it efficiently requires a trick. The trick is related to the choice of the range of σ for the open string: $\sigma \in [0, \pi]$. This may in fact seem as an odd choice since the functions involved are 2π periodic, not π periodic. Before explaining the trick, however, we should derive the equation where it is needed.

The non-zero functional CCR between the fields $X^I(\tau, \sigma)$ and the canonical momenta $\mathcal{P}^{\tau I}(\tau, \sigma)$ above can be written as

$$[X^I(\tau, \sigma), \dot{X}^J(\tau, \sigma')] = 2\pi\alpha' \, i\delta^{IJ} \delta(\sigma - \sigma'). \tag{7.20}$$

Let us emphasise that this CCR cannot be written as a τ derivative on some other CCR since it is an "equal-time" CCR. Acting with a σ derivative gives

$$[X'^I(\tau,\sigma), \dot{X}^J(\tau,\sigma')] = 2\pi\alpha'\, i\delta^{IJ}\partial_\sigma\delta(\sigma-\sigma'). \tag{7.21}$$

Similarly from

$$[X^I(\tau,\sigma), X^J(\tau,\sigma')] = 0, \tag{7.22}$$

we get using σ derivatives

$$[X'^I(\tau,\sigma), X^J(\tau,\sigma')] = 0,\ \ [X^I(\tau,\sigma), X'^J(\tau,\sigma')] = 0,\ \ [X'^I(\tau,\sigma), X'^J(\tau,\sigma')] = 0. \tag{7.23}$$

Finally, the vanishing CCR between two canonical momenta can be written

$$[\dot{X}^I(\tau,\sigma), \dot{X}^J(\tau,\sigma')] = 0. \tag{7.24}$$

With all these CCRs at hand we find that

$$[\dot{X}^I(\tau,\sigma) \pm X'^I(\tau,\sigma), \dot{X}^J(\tau,\sigma') \pm X'^J(\tau,\sigma')] = \pm 4\pi\alpha'\, i\delta^{IJ}\partial_\sigma\delta(\sigma-\sigma'). \tag{7.25}$$

Here we have used the identity $\partial_{\sigma'}\delta(\sigma-\sigma') = -\partial_\sigma\delta(\sigma-\sigma')$. In the last equation the sign choices are all correlated, all up or all down. The other set of sign combinations give

$$[\dot{X}^I(\tau,\sigma) \pm X'^I(\tau,\sigma), \dot{X}^J(\tau,\sigma') \mp X'^J(\tau,\sigma')] = 0. \tag{7.26}$$

Inserting the mode expansions the non-zero commutator above becomes

$$[\dot{X}^I(\tau,\sigma) \pm X'^I(\tau,\sigma), \dot{X}^J(\tau,\sigma') \pm X'^J(\tau,\sigma')] = 2\alpha'\Sigma_{n,m\in Z}e^{-in(\tau\pm\sigma)-im(\tau\pm\sigma')}[\alpha^I_n, \alpha^J_m], \tag{7.27}$$

or, in other words,

$$\Sigma_{n,m\in Z}e^{-in(\tau\pm\sigma)-im(\tau\pm\sigma')}[\alpha^I_n, \alpha^J_m] = \pm 2\pi\, i\delta^{IJ}\partial_\sigma\delta(\sigma-\sigma'). \tag{7.28}$$

Recall now that $\sigma \in [0,\pi]$ which actually seems to create a problem since when expanding the $\delta(\sigma-\sigma')$ in Fourier modes one usually does it using integrals over σ with $\sigma \in [0,2\pi]$:

$$\int_0^{2\pi}\frac{d\sigma}{2\pi}\,e^{in\sigma}e^{im\sigma} = \delta_{n+m,0}, \qquad \delta(\sigma-\sigma') = \Sigma_{n\in Z}\,e^{in(\sigma-\sigma')}. \tag{7.29}$$

We can now understand the reason for using the range $\sigma \in [0,\pi]$ for the open string: The trick mentioned above is to define new functions $A^I(\tau,\sigma)$ which are 2π periodic as follows

$$A^I(\tau,\sigma) = \sqrt{2\alpha'}\Sigma_{n\in Z}\alpha^I_n e^{-in(\tau+\sigma)} := \begin{cases} (\dot{X}^I + X'^I)(\tau,\sigma) \text{ for } \sigma \in [0,\pi], \\ \\ (\dot{X}^I - X'^I)(\tau,-\sigma) \text{ for } \sigma \in [-\pi,0]. \end{cases} \tag{7.30}$$

The four different commutator equations above (for different signs between $\dot{X}^I$ and X'^I) are then summarised in the single equation valid for both σ and σ' in $[0, 2\pi]$

$$[A^I(\tau,\sigma), A^J(\tau,\sigma')] = 4\pi\alpha'\, i\delta^{IJ}\sigma_\sigma\delta(\sigma-\sigma'). \tag{7.31}$$

This implies

$$\Sigma_{n,m\in Z}e^{-in(\tau+\sigma)-im(\tau+\sigma')}\,[\alpha_n^I, \alpha_m^J] = 2\pi\, i\delta^{IJ}\partial_\sigma\delta(\sigma-\sigma'), \tag{7.32}$$

which is the rather trivial to break up in modes. Integrating over $\int_0^{2\pi}\frac{d\sigma}{2\pi}\,e^{in\sigma}\int_0^{2\pi}\frac{d\sigma'}{2\pi}\,e^{im\sigma'}$ gives (after renaming the dummy summation variables above)

$$[\alpha_n^I, \alpha_m^J] = \delta^{IJ}n\,\delta_{n+m,0}. \tag{7.33}$$

We know see that the derivative on the δ function is the reason for the factor of n on the RHS of these CCRs. So finally we understand why we introduced factors $1/\sqrt{n}$ when expanding X^μ in terms of oscillators a_n^μ and $a_n^{\mu\dagger}$ for $n \geq 1$. The relations

$$a_n^I := \frac{1}{\sqrt{n}}\alpha_n^I,\;\; a_n^{I\dagger} := \frac{1}{\sqrt{n}}\alpha_{-n}^I, \tag{7.34}$$

then give rise to the CCR with the standard harmonic oscillator normalisation (no n on the RHS)

$$[a_n^I, a_m^{J\dagger}] = \delta^{IJ}\,\delta_{n,m}. \tag{7.35}$$

It is, however, standard practise in string theory to use the α oscillators!

These CCRs must of course be extended to incorporate also the zero modes. The commutators between zero modes and oscillators vanish which is seen by first integrating the commutator equation $[X^I(\tau,\sigma), \dot{X}^J(\tau,\sigma')] = 2\pi\alpha'\, i\delta^{IJ}\delta(\sigma-\sigma')$ over σ (where both $\sigma, \sigma' \in [0,\pi]$ for this equation to be valid). This gives

$$[x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau, \dot{X}^J(\tau,\sigma')] = 2\alpha' i\delta^{IJ}. \tag{7.36}$$

The σ' dependent terms imply $[x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau, \alpha_n^J] = 0$, and the σ' independent ones imply

$$[x_0^I, p^J] = i\delta^{IJ}. \tag{7.37}$$

In addition we have

$$[x_0^-, p^+] = i\eta^{-+} = -i. \tag{7.38}$$

The world-sheet propagator for the (N,N) open string: (Not in BZ!)
This is good point to make one check of the formalism based on a comparison to ordinary QFT. In QFT a standard step in some calculations is to Wick rotate, i.e., to analytically continue in the time variable, and end up with a theory in Euclidean signature which sometimes can simplify the analysis. In string theory this step has a much more fundamental role since subsequent considerations are then based on complex analysis in one variable.

So, let us take this step here and compute the two-dimensional propagator. Recall the definition

$$\text{Wick rotation: } \tau \to -i\tau \Rightarrow \tau \pm \sigma \to -i(\tau \pm i\sigma). \tag{7.39}$$

Now recall the mode expansion of the (N,N) open string derived in a previous lecture (dropping the spacetime indices since they will play no role here)

$$(N,N): \quad X(\tau,\sigma) = x_0 + 2\alpha' p\tau + i\sqrt{2\alpha'}\Sigma_{n\neq 0}\frac{1}{n}\alpha_n\, e^{-in\tau}\, \cos n\sigma. \tag{7.40}$$

Writing the cos as two exponentials this can be written

$$X(\tau,\sigma) = x_0 + \alpha' p((\tau-\sigma)+(\tau+\sigma)) + i\sqrt{\frac{\alpha'}{2}}\Sigma_{n\neq 0}\frac{1}{n}\alpha_n\,(e^{-in(\tau-\sigma)} + e^{-in(\tau+\sigma)}). \tag{7.41}$$

Now we Wick rotate by replacing $\tau \to -i\tau$ and define complex variables $(z,\bar{z})$:

$$z := e^{\tau - i\sigma},\ \bar{z} := e^{\tau+i\sigma} \Rightarrow \tag{7.42}$$

$$(N,N): \quad X(z,\bar{z}) = x_0 - i\alpha' p \ln z\bar{z} + i\sqrt{\frac{\alpha'}{2}}\Sigma_{n\neq 0}\frac{1}{n}\alpha_n\,(z^{-n} + \bar{z}^{-n}). \tag{7.43}$$

The Euclidean two-point correlation function, the propagator, is obtained as usual in QFT:

$$\begin{aligned}
&{}_x\langle 0|X(z,\bar{z})X(w,\bar{w})|0\rangle_p = \\
&\qquad {}_x\langle 0|(-i\alpha' p\ln z\bar{z} \,+\, i\sqrt{\frac{\alpha'}{2}}\Sigma_{n\geq 1}\frac{1}{n}\alpha_n\,(z^{-n}+\bar{z}^{-n})) \\
&\qquad\qquad (x_0 \,+\, i\sqrt{\frac{\alpha'}{2}}\Sigma_{m\leq -1}\frac{1}{m}\alpha_m\,(w^{-m}+\bar{w}^{-m}))|0\rangle_p.
\end{aligned} \tag{7.44}$$

Here we have used the properties of the two QM vacuum states: $p|0\rangle_p = \alpha_n|0\rangle_p = 0$ for all positive n and ${}_x\langle 0|x_0 = {}_x\langle 0|\alpha_n = 0$ for all negative n. Note also that ${}_x\langle 0|0\rangle_p = 1$. Using the CCR for the oscillators this expression is easily found to be (let $m \to -m$ above)

$$\begin{aligned}
{}_x\langle 0|X(z,\bar{z})X(w,\bar{w})|0\rangle_p &= -i\alpha'\,[p,x_0]\ln z\bar{z} + \frac{\alpha'}{2}\Sigma_{n,m\geq 1}\frac{1}{nm}[\alpha_n,\alpha_{-m}](z^{-n}+\bar{z}^{-n})(w^m+\bar{w}^m) \\
&= -\alpha'\ln z\bar{z} + \frac{\alpha'}{2}\Sigma_{n\geq 1}\frac{1}{n}(z^{-n}+\bar{z}^{-n})(w^n+\bar{w}^n).
\end{aligned} \tag{7.45}$$

Then using the formula $-\ln(1-x) = \Sigma_{n=1}^{\infty}\frac{1}{n}x^n$, for $x<1$, (can be obtained from the geometric series $\frac{1}{1-x} = \Sigma_{n\geq 0}x^n$), this can be written, for $|w|<|z]$,

$${}_x\langle 0|X(z,\bar{z})X(w,\bar{w})|0\rangle_p = -\frac{\alpha'}{2}\left(2\ln z\bar{z} + \ln(1-\frac{w}{z}) + \ln(1-\frac{\bar{w}}{\bar{z}}) + \ln(1-\frac{\bar{w}}{z}) + \ln(1-\frac{w}{\bar{z}})\right). \tag{7.46}$$

Combining the logarithms gives the final answer for the CFT propagator for the (N,N) open string

$${}_x\langle 0|X(z,\bar{z})X(w,\bar{w})|0\rangle_p = -\frac{\alpha'}{2}(\ln(z-w) + \ln(\bar{z}-\bar{w}) + \ln(z-\bar{w}) + \ln(\bar{z}-w)),\ \ |w|<|z]. \tag{7.47}$$

Expressions like this are used heavily in the CFT approach to string theory. We will later present similar expressions for the closed string, both compactified and uncompactified. Note that the logarithms were expected since the Green's function in two spacetime dimensions is obtained from the Fourier transform of $1/p^2$.

The Virasoro and Lorentz algebras:
Finally we have developed the machinery far enough to be able to attack the transverse Virasoro generators and their commutation relations. To derive the resulting Lie algebra, the infinite dimensional Virasoro algebra, is a bit of work but it will be done in detail here. However, the crucial computation is really to use the Virasoro algebra in the derivation and proof of the Lorentz algebra. Although this is one of the most important calculations in the whole subject of string theory it is also one of the most complicated ones unless one develops more advanced CFT techniques. This belongs unfortunately to an advanced string course and will not be done here except for one small step connected to the Polyakov formulation (CFT methods appear in some of the advanced topics for the home project). Thus the final goal of this chapter will be to provide enough details of this proof that we will feel confident that it works.

The transverse Virasoro algebra:
We start by recalling the formulas we will need to derive the Virasoro algebra

$$L_n^\perp = \frac{1}{2}\Sigma_{p\in Z}\, \alpha_{n-p}^I \alpha_p^I, \quad n \in \mathbf{Z} \tag{7.48}$$

$$[\alpha_n^I, \alpha_m^J] = \delta^{IJ} n\, \delta_{n+m,0}, \quad n, m \in \mathbf{Z}. \tag{7.49}$$

The convention is, as already mentioned, to view α_n^I for $n \geq 1$ as annihilation operators and thus α_{-n}^I for $n \geq 1$ as creation operators. Also, α_0^I is as usual related to the momentum p^I. Clearly this implies ordering problems for $L_0^\perp$ but not for the other $L_n^\perp$s. We saw above that the Hamiltonian on the world-sheet $H = L_0^\perp$ so this is not very surprising and is normally dealt with using normal ordering. In string theory the normal ordering constant is given by

$$\frac{1}{2}\Sigma_{n=1}^\infty\, \alpha_n^I \alpha_{-n}^I = \frac{1}{2}\Sigma_{n=1}^\infty\, \alpha_{-n}^I \alpha_n^I + \frac{(D-2)}{2}\Sigma_{n=1}^\infty\, n. \tag{7.50}$$

The reason we cannot just define away the normal ordering constant is that it enters the formula for the mass spectrum of the string as discussed already in a previous lecture. It will also have a fundamental role to play in the proof of the Lorentz symmetry of the string.

We will therefore define $L_0^\perp$ to be normal ordered but add a constant denoted a corresponding to the (perhaps) ill-defined normal ordering constant $\frac{1}{2}(D-2)\Sigma_{n=1}^\infty\, n$ we found above. Thus we replace $L_0^\perp$ by $L_0^\perp + a$ where *from now on* $L_0^\perp$ is normal ordered:

$$L_0^\perp := \frac{1}{2}\alpha_0^I \alpha_0^I + \Sigma_{n=1}^\infty\, \alpha_{-n}^I \alpha_n^I = (L_0^\perp)^\dagger, \tag{7.51}$$

where we have used $(\alpha_n^I)^\dagger = \alpha_{-n}^I$. Note that for $n = 0$ this means that p^I is hermitian since $\alpha_0^I = \sqrt{2\alpha'}p^I$. The relation between p^- and $L_0^\perp$ derived above will now read

$$2\alpha' p^- = \frac{1}{p^+}(L_0^\perp + a), \quad \text{where} \;\; a = \frac{(D-2)}{2}\Sigma_{n=1}^\infty n. \tag{7.52}$$

Then the mass spectrum at the quantum level is determined by the eigenvalues of the operator

$$M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{1}{\alpha'}(L_0^\perp + a) - p^Ip^I = \frac{1}{\alpha'}(\Sigma_{n=1}^\infty \alpha_{-n}^I\alpha_n^I + a). \tag{7.53}$$

This will often be expressed in terms of the number operator $N_n^\perp$ for each transverse set of oscillator pairs, or by the sum of them $N^\perp$, as

$$M^2 = \frac{1}{\alpha'}(\Sigma_{n=1}^\infty N_n^\perp + a) := \frac{1}{\alpha'}(N^\perp + a). \tag{7.54}$$

The importance of the normal ordering constant a is now clear. The hope is then that it can be uniquely determined somehow!

Note that this particular form of this equation is valid for the open string with (N, N) boundary conditions. There will appear several other versions of this formula for other strings later.

Comment: There is a way to obtain a finite number for the infinite sum over all integers using the so called **Riemann zeta-function**

$$\zeta(s) := \Sigma_{n=1}^\infty n^{-s} \quad \Rightarrow \quad \Sigma_{n=1}^\infty n = \zeta(-1) = -\frac{1}{12}. \tag{7.55}$$

This result follows if one analytically continues $\zeta(s)$ in the complex variable s from a region where it is well-defined to the rest of the complex plane. (This result is also discussed in Problem 12.4 in BZ.) Then

$$a = -\frac{D-2}{24}. \tag{7.56}$$

This "strange" result will also follow from the *completely different* calculations below.

We will now derive the Virasoro algebra in a step by step fashion starting from some simple commutators. The first step is

$$[L_m^\perp, \alpha_n^J] = \frac{1}{2}\Sigma_{p\in Z}[\alpha_{m-p}^I\alpha_p^I, \alpha_n^J] = \frac{1}{2}\Sigma_{p\in Z}\alpha_{m-p}^I[\alpha_p^I, \alpha_n^J] + \frac{1}{2}\Sigma_{p\in Z}[\alpha_{m-p}^I, \alpha_n^J]\alpha_p^I. \tag{7.57}$$

Here we use $[\alpha_p^I, \alpha_n^J] = p\delta^{IJ}\delta_{p+n,0}$ and $[\alpha_{m-p}^I, \alpha_n^J] = (m-p)\delta^{IJ}\delta_{m-p+n,0}$. Thus

$$[L_m^\perp, \alpha_n^J] = \frac{1}{2}(-n\alpha_{m+n}^J - n\alpha_{m+n}^J) = -n\alpha_{m+n}^J, \;\; \text{for all} \;\; m, n \in Z. \tag{7.58}$$

This result is clearly correct for $n, m \neq 0$ but is also true for $n = 0$ since p^I commutes with all $L_m^\perp$. The calculation is also correct for $m = 0$ if we keep track of the order of the operators in the calculation which is needed since $L_0^\perp$ is normal ordered.

The next commutator to do is, using $[x_0^I, \alpha_0^J] = \sqrt{2\alpha'} i\delta^{IJ}$, for all $m \in Z$,

$$[L_m^\perp, x_0^J] = -i\sqrt{2\alpha'}\alpha_m^I. \tag{7.59}$$

One feature of these two results is that the integer indices satisfy *conservation of mode number.*

We can now compute the commutator

$$\text{For } m+n \neq 0: \;\; [L_m^\perp, L_n^\perp] = \frac{1}{2}\Sigma_{p\in Z}[L_m^\perp, \alpha_{n-p}^I\alpha_p^I] = \frac{1}{2}\Sigma_{p\in Z}[L_m^\perp, \alpha_{n-p}^I]\alpha_p^I + \frac{1}{2}\Sigma_{p\in Z}\alpha_{n-p}^I[L_m^\perp, \alpha_p^I]. \tag{7.60}$$

Using the above results $[L_m^\perp, \alpha_{n-p}^I] = -(n-p)\alpha_{m+n-p}^I$ and $[L_m^\perp, \alpha_p^I] = -p\alpha_{m+p}^I$, and a shift $p \to p-m$ in the second term so that the two terms can be added, we get

$$\text{For } m+n \neq 0: \;\; [L_m^\perp, L_n^\perp] = \frac{1}{2}\Sigma_p(-(n-p)-(p-m))\alpha_{m+n-p}^I\alpha_p^I = \frac{1}{2}(m-n)\Sigma_p\alpha_{m+n-p}^I\alpha_p^I. \tag{7.61}$$

We thus get back a Virasoro generator and the commutator becomes

$$\text{For } m+n \neq 0: \;\; [L_m^\perp, L_n^\perp] = (m-n)L_{m+n}^\perp. \tag{7.62}$$

It is important in the identification of the RHS that we don't have any ordering issues which is the reason for restricting this calcualtion to $m+n \neq 0$.

We have now done the easy, and in fact classical, part of the Virasoro algebra. If we extend this result by including $m+n=0$ we can write down classical derivative operators that directly generate this classical algebra, which is known as the **Witt algebra**. These generators are

$$V_n := -z^{n+1}\frac{\partial}{\partial z} \;\; \Rightarrow \;\; [V_m, V_n] = (m-n)V_{m+n} \text{ for all } \; n, m \in Z, \;\; z \in \mathbf{C}. \tag{7.63}$$

Mathematically this is the Lie algebra of the group of diffeomorphisms on the circle S^1, i.e., $Lie(Diff(S^1))$.

The QFT version of the Witt Lie algebra is the Virasoro algebra which turns out to have another term on the RHS. The origin of this new term is the operator property of the α_n oscillators and the fact that some special commutators $[L_m^\perp, L_n^\perp]$ will require two α commutators which cannot happen in classical physics (since for Poisson brackets this is impossible). This new term is called the **conformal anomaly** by physicists and the **central extension** by mathematicians. One of its key properties is that it must commute with all the $L_m^\perp$s (thus the name "central"). This requirement comes from the Jacobi identity which any Lie algebra must satisfy (otherwise it cannot be exponentiated to a group).

Taking these facts into account the Virasoro algebra must have the following structure

$$[L^{\perp}_m, L^{\perp}_n] = (m-n)L^{\perp}_{m+n} + A^{\perp}(m)\delta_{m+n,0}, \tag{7.64}$$

where $A^{\perp}$ is the anomaly that we can consider to be a number in the discussion below. The $\delta_{m+n,0}$ multiplying $A^{\perp}$ is put in since we know from above that when $m+n \neq 0$ the new term is not there at all.

Let us first perform the simplest possible calculation that will demonstrate all the points made above. So consider the commutator $[L_2, L_{-2}]$ (dropping "perp" and I indices here so there is only one X component). To see exactly what is going on we organise L_2 as follows

$$L_2 = \frac{1}{2}\Sigma_{n\in Z}\, \alpha_{2-n}\alpha_n = \frac{1}{2}\,\alpha_1\alpha_1 + (\alpha_0\alpha_2 + \alpha_{-1}\alpha_3 + ...), \tag{7.65}$$

using that the terms in the bracket appear twice in the original sum. Similarly we have

$$L_{-2} = \frac{1}{2}\Sigma_{n\in Z}\, \alpha_{-2-n}\alpha_n = \frac{1}{2}\,\alpha_{-1}\alpha_{-1} + (\alpha_{-2}\alpha_0 + \alpha_{-3}\alpha_1 + ...). \tag{7.66}$$

The commutator $[L_2, L_{-2}]$ is then quite easy to compute since each term in L_2 has a non-zero commutator with only one term in L_{-2}. And furthermore, these term-by-term commutators are of only two types 1) involving only terms in the brackets and 2) involving the single term outside the brackets.

1) This computation can be done generally for

$$p \geq 0,\, q \geq 0\; p \neq q: \;\; [\alpha_{-p}\alpha_q, \alpha_{-q}\alpha_p] = \alpha_{-p}[\alpha_q, \alpha_{-q}\alpha_p] + [\alpha_{-p}, \alpha_{-q}\alpha_p]\alpha_q. \tag{7.67}$$

Inserting $[\alpha_q, \alpha_{-q}\alpha_p] = q\alpha_p$ and $[\alpha_{-p}, \alpha_{-q}\alpha_p] = -p\alpha_{-q}$ this becomes

$$p \geq 0,\, q \geq 0\; p \neq q: \;\; [\alpha_{-p}\alpha_q, \alpha_{-q}\alpha_p] = q\alpha_{-p}\alpha_p - p\alpha_{-q}\alpha_q. \tag{7.68}$$

This commutator, involving only a single α commutator, thus gives a normal ordered answer. This is a term that belongs to L_0 on the RHS of the Virasoro algebra, and hence does not contribute to the anomaly $A(2)$.

2) This calculation is a lot more interesting. Let us do it carefully

$$[\frac{1}{2}\alpha_1\alpha_1, \frac{1}{2}\alpha_{-1}\alpha_{-1}] = \frac{1}{4}\alpha_1[\alpha_1, \alpha_{-1}\alpha_{-1}] + \frac{1}{4}[\alpha_1, \alpha_{-1}\alpha_{-1}]\alpha_1 = \frac{1}{2}(\alpha_1\alpha_{-1} + \alpha_{-1}\alpha_1). \tag{7.69}$$

This result is NOT normal ordered so for it to give a term in (the now normal ordered) L_0 we must do a **second** commutation which gives

$$[\frac{1}{2}\alpha_1\alpha_1, \frac{1}{2}\alpha_{-1}\alpha_{-1}] = \alpha_{-1}\alpha_1 + \frac{1}{2}, \tag{7.70}$$

where the first term belongs to L_0 and the second gives the anomaly $A(2) = \frac{1}{2}$. Thus, reinstating the transverse indices on the α oscillators,

$$[L_2^{\perp}, L_{-2}^{\perp}] = 4L_0^{\perp} + \frac{1}{2}(D-2). \tag{7.71}$$

Note that to get $4L_0^{\perp}$ we also need the contributions from 1) above.

Repeating this calculation for other similar cases we get $A(1) = 0$ and $A(3) = 2$. The general case will contain more than one term from "outside the brackets" and leads to

$$A(m) = \frac{1}{2}\Sigma_{n=1}^{m-1}\, n(m-n) = \frac{1}{2}\, m\, \Sigma_{n=1}^{m}\, n - \frac{1}{2}\Sigma_{n=1}^{m}\, n^2, \tag{7.72}$$

where we have added the term $n = m$ (which is 0) to simplify the two sums. These are quite easy to do:

$$\Sigma_{n=1}^{m}\, n = \frac{1}{2}\, m(m+1), \;\; \Sigma_{n=1}^{m}\, n^2 = \frac{1}{6}(2m^3 + 3m^2 + m). \tag{7.73}$$

Thus we get finally $A(m) = \frac{1}{12}m(m^2-1)$. The transverse Virasoro algebra for all X^I components then reads

$$[L_m^{\perp}, L_n^{\perp}] = (m-n)L_{m+n}^{\perp} + (D-2)\frac{m(m^2-1)}{12}\delta_{m+n,0}. \tag{7.74}$$

Note the structure of the conformal anomaly[14]: It is zero for $m = 0, \pm 1$ which means that the subalgebra generated by $L_0^{\perp}, L_{\pm 1}^{\perp}$ is an ordinary Lie algebra, namely $sl(2, \mathbf{R})$, with no anomaly[15]. The conformal anomaly is a pure quantum effect since it originates in the need to perform double commutators which cannot happen in classical physics using Poisson brackets.

Comments: The Virasoro algebra is of enormous importance in many areas of physics and mathematics. It determines most of the basic features of string theory. It also gives the values of all critical exponents in second order phase transitions of systems in two dimensions and hence explains *universality* in two dimensions. The Ising model and many other systems in two dimensions are completely understood by studying the representation theory of the Virasoro algebra.

Before we turn to the proof of the Lorentz algebra we should check how the Virasoro generators act on the string coordinates

$$(N,N): \quad X^{\mu}(\tau,\sigma) = x_0^{\mu} + 2\alpha' p^{\mu}\tau + i\sqrt{2\alpha'}\Sigma_{n\neq 0}\frac{1}{n}\alpha_n^{\mu}e^{-in\tau}\cos n\sigma. \tag{7.75}$$

[14]Quantum generated anomalies also appear in QFT. In the standard model they must be cancelled to keep gauge invariance and hence unitarity. This can be done by choosing the particle spectrum carefully which nature of course has done.

[15] From the Witt algebra we know that this $sl(2,\mathbf{R})$ algebra is generated by $V_{-1} = -\partial_z, V_0 = -z\partial_z, V_1 = -z^2\partial_z$ which can be interpreted as translations, dilatations (scalings) and (special) conformal transformations, respectively, on S^1.

This action is given by the commutator (the p^μ commutes with $L^\perp_m$)

$$[L^\perp_m, X^I(\tau,\sigma)] = [L^\perp_m, x^I_0] + i\sqrt{2\alpha'}\Sigma_{n\neq 0}\frac{1}{n}e^{-in\tau}\cos n\sigma[L^\perp_m, \alpha^I_n]. \tag{7.76}$$

Using the results obtained above for these commutators we get

$$[L^\perp_m, X^I(\tau,\sigma)] = -i\sqrt{\frac{\alpha'}{2}}\Sigma_{n\in Z}(e^{-in(\tau+\sigma)} + e^{-in(\tau-\sigma)})\,\alpha^I_{m+n}. \tag{7.77}$$

To be able to view this as an action on X^I we must express the RHS in terms of X^I. Thus we would like the oscillators on the RHS to be α^I_n, not α^I_{m+n}. This is easily done by shifting the dummy summation variable $n \to n-m$ which gives

$$[L^\perp_m, X^I(\tau,\sigma)] = -\frac{i}{2}\sqrt{2\alpha'}\left(e^{im(\tau+\sigma)}\Sigma_{n\in Z}e^{-in(\tau+\sigma)}\,\alpha^I_n + e^{im(\tau-\sigma)}\Sigma_{n\in Z}e^{-in(\tau-\sigma)}\,\alpha^I_n\right). \tag{7.78}$$

Comparing the sums to the mode expansions of $\dot{X}^I \pm X'^I$ we can write this as

$$[L^\perp_m, X^I(\tau,\sigma)] = -\frac{i}{2}\left(e^{im(\tau+\sigma)}(\dot{X}^I + X'^I) + e^{im(\tau-\sigma)}(\dot{X}^I - X'^I)\right). \tag{7.79}$$

If we separate the τ and σ exponentials this equation reads

$$[L^\perp_m, X^I(\tau,\sigma)] = -ie^{im\tau}\cos m\sigma\dot{X}^I(\tau,\sigma) + e^{im\tau}\sin m\sigma X'^I(\tau,\sigma). \tag{7.80}$$

Although one can interpret this result for any m as coordinate transformations in τ and σ we are primarily interested in what effect $L^\perp_0$ has. Setting $m=0$ gives, with $L^\perp_0 = H$ as we have established previously,

$$[H, X^I(\tau,\sigma)] = -i\partial_\tau X^I(\tau,\sigma). \tag{7.81}$$

This result will have a generalisation in the closed string where it becomes a deep and important new feature of the string as a gauge theory (related to not being able to fix the $\sigma = 0$ point on the string).

Proof of the Lorentz algebra:
Recall the form of the angular momentum two-dimensional currents

$$\mathcal{M}^\alpha_{\mu\nu} := X_\mu\mathcal{P}^\alpha_\nu - X_\nu\mathcal{P}^\alpha_\mu. \tag{7.82}$$

The corresponding open string charges are

$$M_{\mu\nu} = \int_0^\pi (X_\mu\mathcal{P}^\tau_\nu - X_\nu\mathcal{P}^\tau_\mu)d\sigma = \frac{1}{2\pi\alpha'}\int_0^\pi (X_\mu\dot{X}_\nu - X_\nu\dot{X}\mu). \tag{7.83}$$

These charges are conserved (due to Noether's theorem) and hence τ-independent even though the X^μ depend on τ. So, from the two mode expansions only terms with τ-dependence $e^{-in\tau}e^{-im\tau}$ where $m=-n$ will survive. Doing the σ-integral therefore gives the τ-independent result

$$M^{\mu\nu} = x^\mu_0 p^\nu - x^\nu_0 p^\mu - i\Sigma^\infty_{n=1}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n). \tag{7.84}$$

If all the X^μ are quantised independently of each other these Lorentz generators trivially satisfy the Lorentz algebra. This must, however, also be the case in the light-cone gauge (or any other gauge). The problem is that the light-cone calculation of the algebra is not only very lengthy but it is also not automatically satisfied. Instead, as we will explain, it puts constraints on the dimension D of spacetime and the normal ordering constant a introduced above (in connection with $L_0^\perp$).

The M^{IJ} will trivially satisfy the algebra of $SO(D-2)$ since X^I are all independently quantised. The problematic parts of the calculation involve generators containing X^- since these are quadratic in X^I. We will concentrate on the most important commutator $[M^{-I}, M^{-J}]$ which must vanish.

Before starting this calculation we must check that M^{-I} is correctly defined in the quantum theory: it must be hermitian and normal ordered! (It must have real eigenvalues and give zero acting on the vacuum.) From the expression for $M_{\mu\nu}$ above we get

$$M^{-I} = x_0^- p^I - x_0^I p^- - i\Sigma_{n=1}^\infty \frac{1}{n}(\alpha_{-n}^- \alpha_n^I - \alpha_{-n}^I \alpha_n^-), \tag{7.85}$$

which is normal ordered relative $|0\rangle_p$ (check this!) but not hermitian. Then using also $2\alpha' p^- = \frac{1}{p^+}(L_0^\perp + a)$ and $\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp$ (for $n \neq 0$) and making it hermitian by hand, we get

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}(x_0^I L_0^\perp + L_0^\perp x_0^I + 2a x_0^I) - \frac{i}{\sqrt{2\alpha'}p^+}\Sigma_{n=1}^\infty \frac{1}{n}(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp). \tag{7.86}$$

Note that this expression has now become **cubic** in the oscillators α_n^I (in the last term) which makes the computation of the commutator very complicated[16]. So it will not be done here but the extreme importance of the result forces us to give it and see what the implications are. The result is

$$[M^{-I}, M^{-J}] = -\frac{1}{\alpha'(p^+)^2}\Sigma_{n=1}^\infty(\alpha_{-n}^I \alpha_n^J - \alpha_{-n}^J \alpha_n^I)\left(n(1-\frac{D-2}{24}) + \frac{1}{n}(\frac{D-2}{24} + a)\right). \tag{7.87}$$

Then the requirement that $[M^{-I}, M^{-J}] = 0$ implies that the last bracket must vanish for each value of the integer n. This is turn implies the following two conditions

$$1 - \frac{D-2}{24} = 0, \quad \frac{D-2}{24} + a = 0. \tag{7.88}$$

The quite amazing conclusion is then that the quantum relativistic string, i.e., string theory, can only be defined in spacetime dimension $D = 26$ and with the unique value of the normal ordering constant $a = -1$:

$$\text{The bosonic string: } D = 26, \;\; a = -1. \tag{7.89}$$

[16]This calculation can be found in the classic text books "Superstring theory", Vol 1 and 2, by Green, Schwarz and Witten (Cambridge 1988), or using modern CFT techniques in, e.g., "Lecture notes on the bosonic string" by Bengt EW Nilsson (Chalmers).

Note that this result is the same as we obtained from the Riemann zeta-function above: $a := \frac{1}{2}(D-2)\Sigma_{n=1}^{\infty}n = \frac{1}{2}\cdot 24\cdot(-\frac{1}{12}) = -1$. Note also that this result implies the $D = 26$ spacetime mass spectrum

$$M^2 = \frac{1}{\alpha'}(N^{\perp} - 1). \tag{7.90}$$

Recall that

$$N^{\perp} = \Sigma_{n=1}^{\infty}N_n^{\perp} = \Sigma_{n=1}^{\infty}\alpha_{-n}^I\alpha_n^I = \Sigma_{n=1}^{\infty}na_{-n}^I a_n^I. \tag{7.91}$$

This implies that

$$[N^{\perp}, \alpha_{-n}^I] = n\alpha_{-n}^I, \quad \text{for all} \ \ n \in Z. \tag{7.92}$$

The state space:
We are now in a position to start discussing the spectrum of spacetime fields that is contained in the bosonic string. This will be done in terms of the light-cone one-particle states that we discussed before when analysing ordinary field theories in the light-cone gauge. Thus a scalar particle is here represented by a string state $|p^+, p^I\rangle$ which is an eigenstate of the momentum operators p^+ and p^I. This state is also defined to satisfy $\alpha_n^I|p^+, p^I\rangle = 0$ for all $n \geq 1$. So the state $|p^+, p^I\rangle$ is more correctly defined as the tensor product $|p^+\rangle\otimes|p^I\rangle\otimes\Pi_{n\geq 1}|0\rangle_n$ where the ground states $|0\rangle_n$ satisfy $a_n^I|0\rangle_n = 0$ for all $n \geq 1$. Other states are then obtained by acting with the creation operators α_{-n}^I for $n \geq 1$ on the scalar state $|p^+, p^I\rangle$.

Comment: In the field theory obtained as the low-energy approximation of the string (where the string shrinks and becomes point-like) these states are instead obtained by the creation operators from the mode expansion of the fields in Minkowski space, $a^{\dagger}_{(p^+,p^I)}$, acting on the perturbative QFT vacuum state $|0\rangle$. At the level of the free field theory this is not hard to buy but the challenge is to understand how to generate, e.g., the infinite set of interaction terms in $h_{\mu\nu}$ contained in Einstein's theory of gravity given by the Einstein-Hilbert term in GR (after writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$). We will below very briefly return to this crucial issue and describe two different ways to get these interaction terms (none of which can be found in BZ).

To analyse the spectrum we consider the first few states which are

$$|p^+, p^I\rangle, \ \ \alpha_{-1}^I|p^+, p^I\rangle, \ \ \alpha_{-1}^I\alpha_{-1}^J|p^+, p^I\rangle, \ \ \alpha_{-2}^I|p^+, p^I\rangle, ... \tag{7.93}$$

with masses

$$M^2 = \frac{1}{\alpha'}(N^{\perp} - 1) = \frac{1}{\alpha'} \times -1, 0, 1, 1, ... \tag{7.94}$$

This rather strange spectrum calls for some comments:
1) The lowest state is a scalar with negative mass-square, i.e., a tachyon. In Minkowski such a field is associated with a number of problems like instabilities (compare the Higgs field) etc and it would be nice to be able to eliminate it somehow. This will be discussed later.

2) The second set of states corresponds to a massless vector field in $D = 26$ which must come together with a gauge invariance as we have seen in the light-cone analysis of the Maxwell theory. This gauge invariance will be identified later.
3) The third and fourth sets of states are massive (usually of Planck mass size) and give therefore $\frac{1}{2}24 \times 25 + 24 = 324$ states with the same mass. Note that these independent fields are in irreps $(\tilde{IJ}), 1, I$ (where tilde means traceless)[17].
4) After these states there is an infinite set of states with higher and higher masses.

Comment:
The string (and its spectrum) discussed so far is sometimes called **oriented** since it has a sense of direction from the $\sigma = 0$ end to the $\sigma = \pi$ end. It may, however, be of interest to start from the oriented string and construct another string, the **unoriented** one, whose spectrum is a subsector of the oriented one. This new string can be constructed as the sum of two oriented strings which have opposite orientation. This is usually done by introducing an operator Ω that takes σ into $-\sigma$.

Comment on field theory limit: How can it be relevant to discuss string theory in terms of states and fields related to ordinary QFT?
1) At low energies the tension of the string will make it shrink and become more and more point-like when the energy approaches zero.
2) A field theory (effective) action at low energy can be derived (including field theory interactions) in different ways in string theory:
a) using CFT and vertex operators,
b) using conformal invariance and renormalisation group equations for the world-sheet perturbation theory: For the closed string the $\beta = 0$ equations are equivalent to Einstein's and all the other field equations.
More later if time permits!

Comment on the tachyon field theory:
We end this chapter by a discussion of the meaning of the fact that the bosonic string contains a tachyon in the spectrum. This is a quite complicated story so we will be very brief. The tachyon is a scalar quantum field, exactly as a real $\phi(x)$ in QFT, and has creation operators $a^\dagger_{p^+,p^I}$ in the mode expansion that create 1-particle momentum eigenstates when acting on the QFT perturbative vacuum $|0\rangle$. Being tachyonic it satisfies

$$|p^+, p^I\rangle := a^\dagger_{p^+,p^I}|0\rangle, \quad M^2|p^+, p^I\rangle = -\frac{1}{\alpha'}|p^+, p^I\rangle. \tag{7.95}$$

[17]These 324 d.o.f. must have a Lorentz covariant description in field theory in 26 dimensions. It turns out that to find such a formulation one has to introduce so called Stueckelberg fields, see e.g. Appendix A in Park and Lee, hep-th/1908.03704, which leads to a rather complicated set of equations. However, the equations are Stueckelberg symmetric only in $D = 26$ which is consistent with string thyeory.

However, neglecting all other fields (which may be inconsistent to do) the tachyon field $T(x)$ should be described by a Lagrangian of the kind

$$\mathcal{L}(T(x)) = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu T \partial_\nu T - V(T) = \frac{1}{2}\dot{T}^2 - \frac{1}{2}(\nabla T)^2 - V(T), \tag{7.96}$$

where

$$V(T) = \frac{1}{2}M^2 T^2 + \mathcal{O}(T^3). \tag{7.97}$$

So, if the M^2 term is negative the field is a tachyon, at least for fluctuations around $T = 0$.

One of the problematic features of tachyons can be seen clearly by looking at a $T(x)$ field which depends only on time. Then its field equation close to $T = 0$ becomes

$$\ddot{T}(t) + M^2 T(t) = 0 \;\; \Rightarrow \;\; \begin{cases} T(t) = A\cos Mt + B\sin Mt, \;\text{for}\;\; M^2 > 0, \\ \\ T(t) = A\cosh\beta t + B\sinh\beta t, \;\text{for}\;\; M^2 := -\beta^2 < 0. \end{cases} \tag{7.98}$$

Thus we find, as expected, that in a potential bounded from below (like a bowl) the fluctuations are oscillatory while in an unbounded potential (like a bowl turned upside down) the fluctuations will "slide off" the top of the potential which leads to an exponential instability.

The bosonic string must be unstable in a similar sense but the questions are of two kinds:
1) What exactly is the shape of the unbounded potential for the tachyon?
2) If there is a local or global minimum for some expectation value $\langle T \rangle$, what is the theory there?

Although the calculations are rather complicated one has established that the potential is basically of cubic form with a local maximum at $T = 0$ with $V(T = 0) > 0$ and a local minimum at a positive value $T = T^*$ with $V(T = T^*) = 0$. Of course, being basically cubic the potential $V(T)$ runs off to $\pm\infty$ as $T \to \pm\infty$.

The picture, based on the potential described above, that has been adopted by most string theorists is the following:
1) The string theory at the T^* local vacuum is semi-stable.
2) The $D25$ branes have condencated (formed bound states) , and hence disappeared along with all other branes.
3) The last point indicates that the new theory is a closed string but this has not yet been proved[18].

[18] One problem with this interpretation is the closed string tachyon (see the paper from 1999 by Sen and Zwiebach, hep-th/9912249). Note that for superstrings the tachyon can be removed as we will see later.

8 Lectures 8

The purpose of this lecture is to develop the closed string following the steps we took in the previous lectures for the open string. Many of the relevant formulas have already been obtained when solving the wave equation and the constraints in the light-cone gauge. Some of these formulas contain a parameter β which is equal to 2 for the open string but which is now set equal to 1 for the closed string.

8.1 BZ Chapter 13: The relativistic closed string

We start by recalling the formulas relevant for the closed string (using $\beta = 1$): The gauge conditions

$$X^+(\tau,\sigma) = \alpha' p^+ \tau, \;\; \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu, \;\; \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X'^\mu, \tag{8.1}$$

and the solution to the wave equation, and the 2π periodicity,

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu(\tau,\sigma) = 0 \;\; \Rightarrow \;\; X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma) = X^\mu(\tau,\sigma+2\pi). \tag{8.2}$$

Note that the range of σ is different form the open case: The closed string has $\sigma \in [0, 2\pi]$. Since there are no further conditions on $X^\mu(\tau,\sigma)$ we see that it contains **two independent** functions of one variable, now denoted $X_L^\mu(u)$ and $X_R^\mu(v)$ for the *left and right movers*, respectively. Here $u := \tau + \sigma$ and $v := \tau - \sigma$.

The 2π periodicity can be expressed using u and v as

$$X_L^\mu(u+2\pi) - X_L^\mu(u) = X_R^\mu(v) - X_R^\mu(v-2\pi). \tag{8.3}$$

This equation implies that their derivatives $X_L'^\mu$ and $X_R'^\mu$ are both strictly 2π periodic functions and can thus be expanded as

$$X_L'^\mu(\tau+\sigma) = \sqrt{\frac{\alpha'}{2}}\Sigma_{n\in Z}\, \bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}, \tag{8.4}$$

$$X_R'^\mu(\tau-\sigma) = \sqrt{\frac{\alpha'}{2}}\Sigma_{n\in Z}\, \alpha_n^\mu e^{-in(\tau-\sigma)}, \tag{8.5}$$

where the independent oscillators have been distinguished by a bar in the mode expansion for $X_L'^\mu$.

The independence of the unbarred and barred oscillators is certainly true for $n \neq 0$ but what happens for $n = 0$, i.e., the p^μ terms and the corresponding coordinates x_0^μ? To answer this question we start by integrating the above equations. This gives

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}x_{0,L}^\mu + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\Sigma_{n\neq 0}\,\frac{1}{n}\bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}, \tag{8.6}$$

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x_{0,R}^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\Sigma_{n\neq 0}\,\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)}. \tag{8.7}$$

Inserting these expansions into the periodicity condition $X_L(u+2\pi) - X_L(u) = X_R(v) - X_R(v-2\pi)$ implies (only terms linear in u and v survive)

$$\sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^\mu 2\pi = \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu 2\pi \ \Rightarrow \ \alpha_0^\mu = \bar{\alpha}_0^\mu. \tag{8.8}$$

This identification of the zero modes is of course necessary since they correspond to the center of mass momenta. The exact relation is

$$p^\mu := \int_0^{2\pi} \mathcal{P}^{\tau\mu} d\sigma = \frac{1}{2\pi\alpha'}\int_0^{2\pi} \dot{X}^\mu d\sigma = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}}\bar{\alpha}_0^\mu. \tag{8.9}$$

The interesting implication of this trivial fact is that there can be only one canonical set of zero mode coordinates which means that we should set

$$x_0^\mu = x_{0,L}^\mu = x_{0,R}^\mu. \tag{8.10}$$

The final form of the (uncompactified) closed string mode expansion is therefore

$$X^\mu(\tau,\sigma) = x_0^\mu + \alpha' \, p^\mu \, \tau + i\sqrt{\frac{\alpha'}{2}}\Sigma_{n\neq 0}\,\frac{1}{n}(\alpha_n^\mu e^{-in(\tau-\sigma)} + \bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}). \tag{8.11}$$

Note that the coefficient of momentum term in this closed string expansion is different from the one in the open string case. This fact is important in the following comment.

Comment: Repeating the steps that led to the open string world-sheet propagator gives for the (uncompactified) closed string

$$_x\langle 0|X(z,\bar{z})X(w,\bar{w})|0\rangle_p = -\frac{\alpha'}{2}(\ln(z-w) + \ln(\bar{z}-\bar{w})), \ \ |w|<|z|]. \tag{8.12}$$

As for the open case calculation in the light cone gauge, this is valid only for the transverse directions.

For the closed string we have

$$H = \alpha' p^+ p^-, \tag{8.13}$$

and the non-zero CCRs

$$[x_0^-, p^+] = -i, \ \ [x_0^I, p^J] = i\delta^{IJ}, \ \ [\alpha_m^I, \alpha_n^J] = i\delta^{IJ}\delta_{m+n,0}, \ \ [\bar{\alpha}_m^I, \bar{\alpha}_n^J] = i\delta^{IJ}\delta_{m+n,0}. \tag{8.14}$$

We will now return to the solution of the constraints and how to introduce transverse Virasoro generators in the closed string case. The solution to the constraints are (with $\beta = 1$)

$$\dot{X}^- \pm X'^- = \frac{1}{\alpha'}\frac{1}{2p^+}(\dot{X}^I \pm X'^I)^2. \tag{8.15}$$

Since we have different oscillators in the left-moving sector (with mode functions $e^{-in(\tau+\sigma)}$) compared to the right-moving one (with mode functions $e^{-in(\tau-\sigma)}$) we should introduce two different sets of transverse Virasoro generators:

$$(\dot{X}^I + X'^I)^2 = 4\alpha' \Sigma_{n\in Z} \left(\frac{1}{2} \Sigma_{p\in Z} \bar{\alpha}^I_{n-p} \bar{\alpha}^I_p \right) e^{-in(\tau+\sigma)}, \tag{8.16}$$

$$(\dot{X}^I - X'^I)^2 = 4\alpha' \Sigma_{n\in Z} \left(\frac{1}{2} \Sigma_{p\in Z} \alpha^I_{n-p} \alpha^I_p \right) e^{-in(\tau-\sigma)}, \tag{8.17}$$

Defining as usual the sums in the brackets as transverse Virasoro generators we get

$$\dot{X}^- + X'^- = \frac{2}{p^+} \Sigma_{n\in Z}\, \bar{L}^\perp_n e^{-in(\tau+\sigma)}, \tag{8.18}$$

$$\dot{X}^- - X'^- = \frac{2}{p^+} \Sigma_{n\in Z}\, L^\perp_n e^{-in(\tau-\sigma)}. \tag{8.19}$$

This means of course that the closed string is associated with two Virasoro algebras which commute with each other.

Expanding also the X^- fields we get

$$\sqrt{2\alpha'}\bar{\alpha}^-_n = \frac{2}{p^+}\bar{L}^\perp_n, \;\; \sqrt{2\alpha'}\alpha^-_n = \frac{2}{p^+} L^\perp_n. \tag{8.20}$$

These are similar to the open string relation but there is an entirely new feature in the closed string case: Recall that we found above that the momentum modes satisfy $\bar{\alpha}^-_0 = \alpha^-_0$ which implies the so called **level matching condition**

$$L^\perp_0 = \bar{L}^\perp_0, \;\text{ or }\;\; N^\perp = \bar{N}^\perp. \tag{8.21}$$

The second form of the condition follows from $L^\perp_0 = \frac{\alpha'}{4} p^I p^I + N^\perp$ and $\bar{L}^\perp_0 = \frac{\alpha'}{4} p^I p^I + \bar{N}^\perp$ (note that the momenta p^I are the same).

The level matching condition has fundamental implications for the whole closed string theory: But how do we interpret it? And how do we use it?
It is in fact a new kind of condition since it is an equality between two otherwise independent *operators*. It should therefore be used as a condition on state space as we now explain. Recall the mass square operator

$$M^2 = -p^2 = 2p^+p^- - p^I p^I = \frac{2}{\alpha'}(L^\perp_0 + \bar{L}^\perp_0 - 2) - p^I p^I = \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2). \tag{8.22}$$

This operator gives the closed string mass spectrum in terms of the level numbers (eigenvalues of the number operators $N^\perp, \bar{N}^\perp$) which are now demanded to also satisfy the level matching condition $N^\perp = \bar{N}^\perp$. Note that the -2 arises from the normal ordering of the two L_0 operators, each giving -1 as in the open string case.

One can also check that (recall the open string result)

$$[L^\perp_0 + \bar{L}^\perp_0, X^I(\tau,\sigma)] = -i\partial_\tau X^I(\tau,\sigma), \tag{8.23}$$

i.e., the closed string Hamiltonian is $H = L_0^\perp + \bar{L}_0^\perp - 2$, and

$$[L_0^\perp - \bar{L}_0^\perp, X^I(\tau,\sigma)] = i\partial_\sigma X^I(\tau,\sigma). \tag{8.24}$$

The second one is particularly interesting since it means that the σ translation operator $P := L_0^\perp - \bar{L}_0^\perp$ is zero by the level matching condition. Since there is only one mode involved here these σ translations are with a constant parameter. That $P = 0$ can be viewed as a consequence of the fact that we left one small part of the reparametrisation invariance in σ unfixed when choosing the σ gauge, namely the location of the $\sigma = 0$ point on the closed string. We have seen this phenomenon before when computing the Hamiltonian before fixing the τ gauge: We found then that $H = 0$. The reason we make some fuzz about this point here is that there is a more powerful way to handle the quantisation of the string that we will discuss later: The Lorentz covariant Polyakov formulation. There all τ, σ symmetries are left unfixed and then all the Virasoro generators become operator conditions on the space of states just as the level matching did in this light-cone treatment.

The closed string spectrum:
We have now reached one of the central results in this course: The closed string spectrum contains a graviton. Let us discuss some of the lowest levels of the closed string spectrum.

As for the open string also the closed string contains a tachyon as the lowest state

$$|p^+, p^I\rangle, \quad M^2 = -\frac{4}{\alpha'}: \quad T(x) \text{ (tachyon)}. \tag{8.25}$$

The next lowest set of states must satisfy the level matching condition so they are

$$\alpha_{-1}^I \bar{\alpha}_{-1}^J |p^+, p^I\rangle, \quad M^2 = 0: \quad g_{IJ} \text{ (graviton)},\ B_{IJ} \text{ (Kalb-Ramond)},\ \phi \text{ (dilaton)}. \tag{8.26}$$

Note that to satisfy the level matching both α_{-1}^I and $\bar{\alpha}_{-1}^J$ must enter the construction of the state which then has $N^\perp + \bar{N}^\perp = 2$ so that $M^2 = 0$, i.e., we have a number of massless states at this level!. These states have two transversal indices IJ with no symmetries, i.e., the states are $24 \times 24 = 576$ in number. However, as independent fields in QFT they must be split up into irreps of $SO(24)$: $(\tilde{IJ})$, $[IJ]$ and a singlet. They represent, respectively, the graviton 299 dof, 276 dof and one dof.

In a Lorentz covariant field theory description of these fields they will be given by $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ, together with all the interactions between them which are needed to produce a generally covariant theory in $D = 26$ spacetime dimensions. Also $B_{\mu\nu}$ turns out to be a field with gauge transformation $\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$. The field theory therefore will be formulated in terms of a field strength with three antisymmetric indices, a direct generalisation of the Maxwell theory:

$$H_{\mu\nu\rho} := 3\partial_{[\mu} B_{\nu\rho]}. \tag{8.27}$$

Such a field has interesting physical use as axions in $D = 4$ spacetime dimensions and is, e.g., considered as a **dark matter** candidate.

String interactions: The dilaton ϕ has a key role to play: It is related to the coupling constant in string theory.
Consider first the metric field $g_{\mu\nu}$ and how it enters the theory of general relativity. In the discussion of gravitational waves we are used to expand $g_{\mu\nu}$ in terms of small fluctuations $h_{\mu\nu}$ around some background geometry $\langle g_{\mu\nu}\rangle$, i.e. we have

$$g_{\mu\nu}(x) = \langle g_{\mu\nu}\rangle(x) + \kappa h_{\mu\nu}(x). \tag{8.28}$$

Here we have introduced a parameter κ as $8\pi G_N = \kappa^2$ so that the Einstein-Hilbert action at the lowest bilinear level gives just a kinetic term for $h_{\mu\nu}$ normalised with a factor $\frac{1}{2}$ as is standard for real fields. Note that the dimension of κ is $[\kappa] = L$ which forces $h_{\mu\nu}$ to have dimension L^{-1}. This is a very general situation where $\langle g_{\mu\nu}\rangle$ can be any maximally symmetric **background** geometry, Minkowski, de Sitter (dS) or anti de Sitter (AdS).

Expanding the Einstein-Hilbert action beyond the first bilinear term shows that κ plays the role of coupling constant in a gravity theory. Terms with all higher powers of $h_{\mu\nu}$ will appear and they will all have two derivatives. Schematically (not writing out the indices) the interaction terms are of the form

$$\mathcal{L}(h_{\mu\nu}) = \frac{1}{2}\partial h \partial h + \kappa h(\partial h)^2 + (\kappa h)^2(\partial h)^2 + ... \tag{8.29}$$

One can check that this means that a standard Feynman graph loop expansion a g-loop diagram comes with a factor $(\kappa)^{2g}$. If we want to understand this result in string theory we should at least identify the gravitational coupling constant and see if it behaves in the same way as κ.

Note: This discussion is very dimension dependent. In D spacetime dimensions the dimension of κ is $[\kappa_D^2] = L^{D-2}$.

The important question is: How is κ obtained in string theory? The only parameter we have encountered in string theory so far is α' which has dimension L^2. Thus we expect a relation like $\kappa_D^2 \sim (\alpha')^{\frac{D-2}{2}}$ but α' is not related to how strings interact with each other or to the string loop expansion. Where do we find the relation to the loop expansion?

The answer is quite surprising: The **closed string coupling constant**, which is also the loop counting parameter, comes from the dilaton. The background expansion for the metric above can be carried over to only one other kind of field, namely scalars. We are familiar with this from the Higgs field. Thus we can write the dilaton as a fluctuation relative a constant background value

$$\phi(x) := \langle\phi\rangle + \varphi(x). \tag{8.30}$$

We now claim that the background value (VEV) of the dilaton, $\langle\phi\rangle$, defines the closed string coupling constant g_s by

$$g_s := e^{\langle\phi\rangle}. \tag{8.31}$$

Comment: This fact can be understood by looking at how the string path integral behaves at higher string loops. We will provide a very brief discussion of the path integral after having introduced the covariant Polyakov action.

From this string path integral discussion we will also find the g_s dependence of the Newton constant which in bosonic string theory reads

$$G_N^{(26)} \sim g_s^2(\alpha')^{12}. \tag{8.32}$$

The appearance of g_s^2 in this formula is in fact very general as will be clear from the path integral later.

Since the superstring lives in $D = 10$ spacetime dimensions the corresponding equation is

$$G_N^{(10)} \sim g_s^2(\alpha')^4. \tag{8.33}$$

In terms of the Planck length l_P and the string length $l_s = \sqrt{\alpha'}$, this becomes

$$l_P^{(10)} \sim l_s g_s^{\frac{1}{4}}. \tag{8.34}$$

Using now the results discussed previously in connection with compactifications from $D = 10$ to $D = 4$ dimensions we find that Newton's constant in $D = 4$ is given by

$$G_N = \frac{G_N^{(10)}}{V^{(6)}} \sim \frac{g_s^2(\alpha')^4}{V^{(6)}} = \frac{g_s^2\alpha'}{V^{(6)}/(\alpha')^3}. \tag{8.35}$$

So if the dimensionless ratio $V^{(6)}/(\alpha')^3$ is fixed and close to 1, then the Newton's constant in $D = 4$ spacetime dimensions is

$$G_N \sim g_s^2\alpha'. \tag{8.36}$$

There is another amazing relation between coupling constants in string theory: If one introduces also an open string coupling constant, g_o, (needed since two open strings can join at the ends) then one can argue that

$$g_o^2 \sim g_s. \tag{8.37}$$

Comment: As for the previous comment also the following one explaining this relation between coupling constants is best carried out after the introduction of the Polyakov action and its conformal world-sheet symmetry. The crucial property here is the conformal symmetry in two dimensions which means that one can deform any string world-sheet any way one wants. So, consider two open string strings approaching each other and then joining at the ends to form a single open string. This 3-point interaction should be associated with a coupling constant g_o. If this third string propagates a short distance and then splits up into two (a new factor of g_o) which then join again (another factor of g_o), and finally the last open string splits in two (a new factor g_o) we have produced a one-loop correction to the open string 2 to 2 scattering. This diagram has a factor g_o^4. But, using the conformal invariance we can pull the circle in the middle out of the surface of the open string diagram which then becomes a closed string. The whole diagram must in this new interpretation be a tree diagram scattering of a closed string and two open strings with coupling constant $g_o^2 g_s$. We should also note that since the open string contains vector gauge fields g_o will correspond to the Yang-Mills coupling constant g_{YM} in the low energy field theory limit of the open string. Thus

$$g_s \sim g_{YM}^2. \tag{8.38}$$

We now leave this very important discussion of the connection between parameters in field theory G_N, g_{YM} and the ones in string theory α', g_s, g_o. We will return to these relations again in the context of AdS/CFT. Here we will instead continue with a discussion of orbifolds in string theory.

Orbifold string mode expansions:

In field theory one should not use background geometries with singular points (or surfaces) like the tip of a cone. The reason is that this leads to infinities like the one at the center of a black hole or the position of a point charge. As we will see below string theory can handle this situation in a natural way.

For this purpose we consider a *target spacetime* using light-cone coordinates (x^+, x^-, x^I) and impose a $\mathbf{Z}_2$ identification acting in the 25th direction:

$$x^\mu = (x^+, x^-, x^I) =: (x^+, x^-, x^i, x^{25}) \tag{8.39}$$

$$\text{with } \mathbf{Z}_2 \text{ action}: \quad x^\mu = (x^+, x^-, x^i, x^{25}) \to \tilde{x}^\mu := (x^+, x^-, x^i, -x^{25}). \tag{8.40}$$

The result of a $\mathbf{Z}_2$ identification $x^\mu \sim \tilde{x}^\mu$ is a fundamental region given by the half-space $x^{25} \geq 0$ where the "fix-point" $(x^+, x^-, x^i, x^{25} = 0)$ is singular. The first important observation is that there are now *two kinds of closed string configurations*:

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) \quad \text{and} \quad X^\mu(\tau, \sigma + 2\pi) = \tilde{X}^\mu(\tau, \sigma), \tag{8.41}$$

where the second case gives a closed string due the identification. We will see that this means that the closed string on an orbifold will have two sectors, one **untwisted** (case 1 above) and one **twisted** (case 2 above).

The untwisted sector consists of all closed strings that are *invariant* under the $\mathbf{Z}_2$ identification. In this sector all components $X^\mu(\tau, \sigma)$ have the same closed string mode expansion as discussed previously. To check what the invariance under $\mathbf{Z}_2$ means it is useful to introduce a unitary operator U that generates the $\mathbf{Z}_2$ transformation:

$$\mu \neq 25: \quad UX^\mu U^{-1} = X^\mu, \quad \mu = 25: \quad UX^{25}U^{-1} = -X^{25}. \tag{8.42}$$

Then since $U|p^+, p^i, p^{25}\rangle = |p^+, p^i, -p^{25}\rangle$ the invariant ground state is

$$|p^+, p^i, p^{25}\rangle + |p^+, p^i, -p^{25}\rangle. \tag{8.43}$$

At level 1 (i.e., $N^\perp = \bar{N}^\perp = 1$) the invariant states are of four kinds:

$$\alpha^i_{-1}\bar{\alpha}^j_{-1}(|p^+, p^i, p^{25}\rangle + |p^+, p^i, -p^{25}\rangle), \tag{8.44}$$
$$\alpha^i_{-1}\bar{\alpha}^{25}_{-1}(|p^+, p^i, p^{25}\rangle - |p^+, p^i, -p^{25}\rangle), \tag{8.45}$$
$$\alpha^{25}_{-1}\bar{\alpha}^j_{-1}(|p^+, p^i, p^{25}\rangle - |p^+, p^i, -p^{25}\rangle), \tag{8.46}$$
$$\alpha^{25}_{-1}\bar{\alpha}^{25}_{-1}(|p^+, p^i, p^{25}\rangle + |p^+, p^i, -p^{25}\rangle). \tag{8.47}$$

Note that the Hamiltonian is also invariant (since quadratic in X^μ): $UHU^{-1} = H$. This implies that it is consistent to restrict the theory to the untwisted sector. However, for reasons related to properties of strings at 1-loop the twisted sector (below) must be included to make the theory consistent.

The twisted sector consists of all string configurations that need the identification for the string to become closed. Thus the mode expansion of $X^{\mu\neq 25}$ is the standard one but the one for X^{25} is different. It is anti-periodic:

$$X^{25}(\tau, \sigma + 2\pi) = -X^{25}(\tau, \sigma). \tag{8.48}$$

Using the split into left-movers and right-movers we have

$$X'^{25}_L(u + 2\pi) = -X'^{25}_L(u), \quad X'^{25}_R(v - 2\pi) = -X'^{25}_R(v). \tag{8.49}$$

This anti-periodicity changes the mode expansion completely. Instead of the usual one we now get one in terms of half-integers $r \in \mathbf{Z} + \frac{1}{2}$ (suppressing 25):

$$X_L(u) = x_{0,L} + i\sqrt{\frac{\alpha'}{2}}\Sigma_{r\in\mathbf{Z}+\frac{1}{2}}\frac{1}{r}\bar{\alpha}_r e^{-iru}, \tag{8.50}$$

$$X_R(v) = x_{0,R} + i\sqrt{\frac{\alpha'}{2}}\Sigma_{r\in\mathbf{Z}+\frac{1}{2}}\frac{1}{r}\alpha_r e^{-irv}, \tag{8.51}$$

Implementing the anti-periodicity gives $x_0 = x_{0,L} + x_{0,R} = 0$ which also means that the center of mass momentum vanishes. This is, however, a trivial statement since there is no momentum mode when using sums over half-integers (all modes multiply exponentials so there are no zero modes):

$$X(\tau, \sigma) = i\sqrt{\frac{\alpha'}{2}}\Sigma_{r\in\mathbf{Z}+\frac{1}{2}}\frac{1}{r}\left(\bar{\alpha}_r e^{-ir(\tau+\sigma)} + \alpha_r e^{-ir(\tau-\sigma)}\right). \tag{8.52}$$

Note: A crucial consequence of this mode expansion (without any zero modes) is that these strings are tied to the fix-point $x^{25} = 0$!

To obtain the effect of this half-integer expansion on the mass spectrum we need to recompute the Virasoro generator $L^\perp_0$. From the definition

$$L^\perp_0 = \frac{1}{2}\Sigma_{p\in\mathbf{Z}}\alpha^i_{-p}\alpha^i_p + \frac{1}{2}\Sigma_{r\in\mathbf{Z}+\frac{1}{2}}\alpha^{25}_{-r}\alpha^{25}_r, \tag{8.53}$$

we get a new normal ordering constant

$$a = \frac{1}{2}(D-3)\Sigma_{p=1}^{\infty} p + \frac{1}{2}\Sigma_{r=\frac{1}{2}}^{\infty} r. \tag{8.54}$$

As before the sum over the integers in the first term gives $-\frac{1}{12}$ while the second sum is computed as follows

$$\Sigma_{r=\frac{1}{2}}^{\infty} r = \tfrac{1}{2} + \frac{3}{2} + \frac{5}{2} + ... = \frac{1}{2}\Sigma_{p=1,3,5,..} p = \frac{1}{2}\Sigma_{p=1}^{\infty} p - \frac{1}{2}\Sigma_{p=2,4,6,...} p = \tag{8.55}$$

$$\frac{1}{2}\Sigma_{p=1}^{\infty} p - \Sigma_{p=1}^{\infty} p = \frac{1}{2}(-\frac{1}{12}) - (-\frac{1}{12}) = \frac{1}{24}. \tag{8.56}$$

Then since for the bosonic string we still have $D = 26$ we find for each left and right moving sector

$$a = \frac{1}{2}(D-3)\left(-\frac{1}{12}\right) + \frac{1}{2}\left(\frac{1}{24}\right) = -\frac{15}{16}. \tag{8.57}$$

Inserting this into the formula for M^2 it becomes

$$\text{Twisted sector: } M^2 = \frac{2}{\alpha'}\left(N^{\perp} + \bar{N}^{\perp} - \frac{15}{8}\right), \quad N^{\perp} = \bar{N}^{\perp}. \tag{8.58}$$

Thus there are tachyons but no massless fields in this sector.

9 Lecture 9

At this point in the development of string theory it would be very useful to leave the light-cone formulation and express the bosonic string in a Lorentz covariant and more powerful way. This will also help us to take the rather huge step over to the superstring. In addition we can follow up on the role of the dilaton and the closed string coupling constant mentioned previously. We will follow BZ Chap. 24 with some minor excursions. Doing this will also provide some insight into how modern string theory is handled mathematically. Unfortunately, some of the material is rather tricky so some comments are made only as hints towards the more advanced and modern treatment of string theory.

9.1 Chapter 24: The covariant quantisation of the bosonic string.

Recall the starting point, i.e., the Nambu-Goto action

$$S[X] = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-\gamma}, \quad \gamma := \det\gamma_{\alpha\beta}, \tag{9.1}$$

where $\gamma_{\alpha\beta}$ is the pull-back metric from the target spacetime, here Minkowski,

$$\gamma_{\alpha\beta} := \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{9.2}$$

To quantise the Nambu-Goto action in a covariant manner is tricky and should be avoided if possible. Fortunately, there is a way to express string theory in terms of an action based on **free** X^μ fields. Then **all components** of X^μ can be expanded in modes which are quantised independently of each other. This rather amazing step leads to the Polyakov action

$$S[X^\mu, h_{\alpha\beta}] = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-h}\, h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{9.3}$$

The huge differences compared to the Nambu-Goto action are the absence of the square root involving the string coordinates X^μ and the presence of a new world-sheet field $h_{\alpha\beta}(\tau,\sigma)$. This field plays the role of a metric on the world-sheet and is independent of the pull-back metric on the world-sheet $\gamma_{\alpha\beta}(\tau,\sigma)$. The Polyakov action above is thus ("trivially" as in GR) reparametrisation invariant on the world-sheet just as the Nambu-Goto action is but for a different reason. Note that $h := \det h_{\alpha\beta}(\tau,\sigma)$. The new action is also trivially invariant under global Poincaré transformations.

There is, however, one more crucial difference between the two actions since the Polyakov has one more local symmetry, (manifest) local scale invariance, also known as **Weyl invariance**:

$$\text{Weyl transformations:} \;\; h_{\alpha\beta}(\tau,\sigma) \to e^{2\Omega(\tau,\sigma)}h_{\alpha\beta}(\tau,\sigma), \;\; X^\mu \to X^\mu. \tag{9.4}$$

The reason for claiming that the string can be described by the Polyakov action instead of the Nambu-Goto is that they give rise to the same dynamics, i.e., that they are classically **equivalent on-shell**. This fact is rather easy to prove: We must just show that the field equations are the same. So, if we can show that the two Lagrangians are equal **on-shell** we are home.

Let us start from the Polyakov action. It contains two independent fields on the world-sheet, $h_{\alpha\beta}(\tau,\sigma)$ and $X^\mu(\tau,\sigma)$, and hence gives rise to two field equations: the two-dimensional covariant Klein-Gordon equation

$$\delta X^\mu(\tau,\sigma) \Rightarrow \Box_2 X^\mu = 0, \tag{9.5}$$

where $\Box_2 := \nabla^\alpha(h)\nabla_\alpha(h)$, and the two-dimensional Einstein equations

$$\delta h^{\alpha\beta} \Rightarrow 0 = T_{\alpha\beta}, \quad T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2} h_{\alpha\beta}(h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu}). \tag{9.6}$$

Note that the left hand side is zero since the action has no Einstein-Hilbert term in it. In fact, even if one adds a two-dimensional Einstein-Hilbert term to the Polyakov action it would not change the Einstein equations since $\int d\tau d\sigma\sqrt{-h}R_2(h)$ is a total derivative and contributes only to the boundary terms (recall that the field equations come from the bulk term in $\delta S = 0$). We will come back to the Einstein-Hilbert term shortly when returning to the discussion of the string coupling g_s.

The Einstein equations above can be written

$$\gamma_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}(h^{\gamma\delta}\gamma_{\gamma\delta}). \tag{9.7}$$

Taking the determinant of this equations gives $(\det\gamma_{\alpha\beta}) = \frac{1}{4}(\det h_{\alpha\beta})(h^{\gamma\delta}\gamma_{\gamma\delta})^2$ that is

$$\sqrt{-\gamma} = \frac{1}{2}\sqrt{-h}(h^{\gamma\delta}\gamma_{\gamma\delta}). \tag{9.8}$$

This equation proves that on-shell the two Lagrangians are equal (the LHS is the Nambu-Goto Lagrangian and the RHS the Polyakov one).

Having established that the two actions give the same physics, we will now study the string using the Polyakov action and applying a covariant quantisation formalism. Some of the steps discussed below will be rather sketchy for lack of space in this course. To define the starting point we have the two field equations

$$\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu) = 0, \tag{9.9}$$

$$\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2} h_{\alpha\beta}(h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu}) = 0. \tag{9.10}$$

The first thing to do is to choose the so called **conformal gauge**

$$h_{\alpha\beta} = \rho^2(\tau,\sigma)\eta_{\alpha\beta}. \tag{9.11}$$

This is possible because we can just use the two reparametrisation invariances on the world-sheet to impose this gauge condition. Recall the harmonic gauge in ordinary GR which is used there in a similar way. In the present case the conformal gauge has some very nice consequences. Inserting it into the field equations gives

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\mu = 0, \tag{9.12}$$

$$\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}(\eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu}) = 0. \tag{9.13}$$

This nice simplification is an effect of the Weyl invariance mentioned above (the scale factor ρ^2 cancels in both equations even though it is a function of the world-sheet coordinates).

The first equation is just the wave equation again, but what is the second set of (three) equations? Choose first $\alpha = \beta = \tau$. Then it reads

$$(\dot{X}^\mu)^2 + \frac{1}{2}(-(\dot{X}^\mu)^2 + (X'^\mu)^2) = 0 \Rightarrow \dot{X}^2 + X'^2 = 0. \tag{9.14}$$

The same result is obtained by choosing $\alpha = \beta = \sigma$ while choosing one index as τ and one as σ we get $\dot{X}^\mu X'_\mu = 0$. As we did in the light-cone situation these constraints can be summarised as

$$(\dot{X}^\mu \pm X'^\mu)^2 = 0. \tag{9.15}$$

So, the Polyakov action has done a very good job so far (by obtaining the constraints from the world-sheet Einstein's equations).

The question now is how to quantise the theory with the above constraints while keeping the Lorentz invariance and the independence of all the components of X^μ? Note that, contrary to the light-cone gauge, we have not yet imposed any gauge conditions on any components of X^μ!

This is where the new situation must be made clear. With all components of X^μ quantised independently the Lorentz algebra is trivially satisfied. The conditions on the dimension D of spacetime and on the normal ordering constant a can therefore not be obtained by demanding that the Lorentz generators satisfy the Lorentz algebra after quantisation as we did in the case of the light cone gauge.

A second problem is that, since all the oscillators in the time components remain, the creation operators in X^0 will give rise to states with negative norm (which will ruin unitarity).

In order to resolve these problems we start by trying to impose $(\dot{X}^\mu \pm X'^\mu)^2 = 0$ as an **operator condition** on the space of states generated by the creation operators α^μ_{-n}. This may be viewed as a generalisation of the level matching condition $L_0^\perp = \bar{L}_0^\perp$. To do this we expand the covariant constraint in a new set of covariant Virasoro generators now containing all X^μ components (open string):

$$(\dot{X}^\mu \pm X'^\mu)^2 := 4\alpha' \Sigma_{n\in\mathbf{Z}}\, L_n e^{-in(\tau\pm\sigma)}, \tag{9.16}$$

where

$$L_n := \frac{1}{2}\Sigma_{p\in\mathbf{Z}} :\alpha^{\mu}_{n-p}\,\alpha^{\nu}_{p}:\,\eta_{\mu\nu}. \tag{9.17}$$

This means that there is a normal ordering constant a coming from the covariant L_0 whose value must be determined by some new covariant methods. Formally, of course, this covariant constant is given by

$$a := \frac{D}{2}\,\Sigma_{n=1}^{\infty}\, n. \tag{9.18}$$

Thus, at this point it is far from clear how to find the result $a = -1$ in the covariant formulation.

Before sketching the answer to this question, we should make sure that the present situation makes sense. The first thing to check is if there are symmetries left unfixed corresponding to all the L_n generators (similar to the constants shifts in τ and σ generated by $L_0^{\perp}$ as we have seen above in the light-cone case)?

To see that the answer is Yes to this question we note that having fixed the world-sheet metric to $h_{\alpha\beta}(\tau,\sigma) = \rho^2(\tau,\sigma)\eta_{\alpha\beta}$ we can use the Weyl invariance to fix it to be exactly flat: $h_{\alpha\beta}(\tau,\sigma) = \eta_{\alpha\beta}$. Then we should ask what symmetries remain by solving $\delta h_{\alpha\beta}(\tau,\sigma) = 0$. This equation becomes, due to the local reparametrisation and Weyl invariances,

$$\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha + f\,\eta_{\alpha\beta} = 0, \tag{9.19}$$

where the parameters of the local reparametrisation and Weyl invariances are, respectively, $\xi_\alpha(\tau,\sigma)$ and $f(\tau,\sigma)$. Writing these equations in light-cone variables $\sigma^{\pm} := \tau \pm \sigma$ gives

$$\partial_+\xi_+ = 0, \tag{9.20}$$

$$\partial_-\xi_- = 0, \tag{9.21}$$

$$\partial_+\xi_- + \partial_-\xi_+ = f. \tag{9.22}$$

The last equation just determines $f(\tau,\sigma)$ and is therefore not a condition on $\xi_\alpha(\tau,\sigma)$. The other two equations, $\partial_+\xi_+ = 0$ and $\partial_+\xi_+ = 0$, imply (not the position of the indices $\pm$)

$$\xi^+(\tau,\sigma) = \xi^+(\sigma^+), \quad \xi^-(\tau,\sigma) = \xi^-(\sigma^-). \tag{9.23}$$

The symmetries that remain after the conformal gauge is imposed is therefore given by the two arbitrary functions $(\xi^+(\sigma^+),\xi^-(\sigma^-))$. After a Wick rotation to Euclidean signature on the world-sheet this symmetry corresponds to holomorphic transformations, or in other words, conformal transformations. This means that in principle we could still remove the time component of X^μ (also satisfying $\Box_2 X^\mu = 0$) by a further gauge condition on the coordinates (σ^+,σ^-). Since we want to stay Lorentz covariant this will, however, not be done.

The second issue concerns the use of the (now covariant) Virasoro generators L_n. The strategy is to implement $L_n = 0$, for all $n \in \mathbf{Z}$, by imposing it on the space of states.

Denote an arbitrary state as $|\Phi\rangle$ and assume for now that $a = -1$. Then the condition for $n = 0$ becomes

$$0 = (L_0 - 1)|\Phi\rangle = (\frac{1}{2}\alpha_0^\mu \alpha_{0\mu} + \Sigma_{p=1}^{\infty} \alpha_{-p}^\mu \alpha_{p\mu} - 1)|\Phi\rangle. \tag{9.24}$$

Since $\alpha_0^\mu \alpha_{0\mu} = \frac{2}{\alpha'} p^2$ this condition can be reformulated as

$$M^2 = -p^2 = \frac{1}{\alpha'}(N-1) \text{ acting on } |\Phi\rangle. \tag{9.25}$$

Consider now the generators L_n for $n \neq 0$. The Virasoro algebra is derived in precisely the same way as for the transverse $L_n^\perp$. Then

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12} m(m^2-1)\, \delta_{m+n,0}. \tag{9.26}$$

Note that since L_n contains all components of X^μ the central term contains a factor D (instead of $D-2$ in the light-cone).

Imposing all $L_n = 0$ for $n \neq 0$ implies, however, for instance for $n = 2$

$$[L_2, L_{-2}]|\Phi\rangle = 0 \Rightarrow 0 = 4 + D\frac{2 \cdot 3}{12}, \tag{9.27}$$

which obviously is not a sensible result since D comes out negative. So, imposing all $L_n = 0$ for all $n \neq 0$ does not work!

Fortunately, the operator condition $L_n = 0$ for all $n \neq 0$ can be implemented in a less constraining way by recalling that operator conditions should in general be interpreted as conditions on matrix elements: Impose $L_n|\Phi\rangle = 0$ only for $n > 0$! But this then means that the conjugated equations $\langle\Phi|L_n = 0$ for $n < 0$ are also imposed. Thus, between two such states we have

$$\langle\Phi|L_n|\Phi'\rangle = 0, \text{ for all non-zero } n \in \mathbf{Z}. \tag{9.28}$$

Now we know how to handle the Virasoro generators and how to use them to constrain the space of states generated by the creation operators α_{-n}^μ for $n > 0$. We will come back to the structure of this state space below but first we should provide arguments for $D = 26$ and $a = -1$. Classically the Polyakov string is conformal (Weyl) invariant. We will now **demand** this to be true also at the quantum level which means that the conformal anomaly must vanish.

This is possible to achieve due to the following fact: Since in the covariant treatment of X^μ also the time component appears in all *loop calculations on the world-sheet*. So its bad consequences (ruining unitarity) must be eliminated somehow. This is done by the so called Faddeev-Popov procedure: The remaining gauge symmetries imply that so called

anti-commuting ghost fields, here denoted (b, c), must introduced. Since they are anti-commuting they can be designed to exactly cancel out all effects of X^0. (Compare this to why supersymmetry is interesting.)

For the covariant string this means that we must add ghost terms to the Polyakov action which then give contributions to the stress tensor and hence to L_n. They now become

$$L_n := L_n^{(X)} + L_n^{(b,c)} \Rightarrow \; c = c^{(X)} + c^{(b,c)} = D - 26 = 0 \Rightarrow D = 26. \tag{9.29}$$

To find $a = -1$ in the covariant theory is more tricky. Let us very briefly mention three ways to argue for the value $a = -1$:
1) Conformal invariance implies that one must be able to integrate vertex operators over the world-sheet.
2) Interactions demand that the scattering of two physical states produce a third physical state.
3) Gauge invariance in the low energy field theory only works if $D = 26$ and $a = -1$.

Let us be a bit more explicit about the last issue. To do that we need to introduce some more structure in connection with the Virasoro algebra above. The general form of the Virasoro algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\,\delta_{m+n,0}, \tag{9.30}$$

where c is a number, the conformal anomaly (or central extension), that defines this version of the algebra. Representations of this algebra are defined from a **lowest weight state** $|c, h\rangle$ by

$$L_0|c,h\rangle \,= h|c,h\rangle, \tag{9.31}$$

$$L_n|c,h\rangle \,= 0, \;\; n \geq 1. \tag{9.32}$$

The representation (called a Verma module) is infinite dimensional since any combinations of L_{-n} for $n \geq 1$ acting on the ground state $|c, h\rangle$ is a new state in the representation denoted (c, h). Then the state $|c, h\rangle$ is called *primary* and states obtained from $L_{\{-n\}}$ acting on it are called *descendent* states. Here $\{-n\}$ denotes any selection of a set of negative mode numbers. In string theory the states $|c, h\rangle$ are called **physical** and sometimes denoted $|\Phi\rangle$ or just $|phys\rangle$.

With these definitions there may appear states $|\chi\rangle$ that are special in the sense that they are primary and descendent at the same time. Such states satisfy

$$L_n|\chi\rangle = 0, \text{ for all } n > 0, \;\text{ AND }\; |\chi\rangle = L_{\{-n\}}|\tilde{\chi}\rangle \text{ for some } |\tilde{\chi}\rangle \text{ and } \;\{-n\}. \tag{9.33}$$

States $|\chi\rangle$ of this kind are orthogonal to all states including themselves, i.e., for any such state $|\Phi\rangle$,

$$\langle\Phi|\chi\rangle = 0, \;\; \langle\chi|\chi\rangle = 0. \tag{9.34}$$

Thus $|\chi\rangle$ is often called a **null state** and can be used to make a shift of a state $|\Phi\rangle$ that has no physical consequences (much like a gauge transformation in field theory):

$$|\Phi\rangle \sim |\Phi'\rangle = |\Phi\rangle + |\chi\rangle. \tag{9.35}$$

Let us consider a relevant example involving null states. Consider a physical state in the open string

$$|\Phi\rangle := \xi_\mu \alpha^\mu_{-1}|p^\mu\rangle. \tag{9.36}$$

This state can be identified with a one-particle state in QED obtained by applying a creation operator $a^{\mu\dagger}_{\mathbf{p}}$ from the mode expansion of the Maxwell field $A^\mu(x)$. ξ_μ is then the corresponding polarisation tensor. However, for this interpretation to be OK this state must satisfy $p^2 = 0$ and $p^\mu \xi_\mu = 0$. In the string construction of this state these conditions come from the physical state requirements. Consider first $L_0 = 1$:

$$L_0(\xi_\mu \alpha^\mu_{-1}|p^\mu\rangle) = \xi_\mu \alpha^\mu_{-1}|p^\mu\rangle \Rightarrow p^2 \xi_\mu \alpha^\mu_{-1}|p^\mu\rangle = 0, \tag{9.37}$$

where we have used that $L_0 = \alpha' p^2 + N$. As a second condition we have $L_1 = 0$:

$$L_1(\xi_\mu \alpha^\mu_{-1}|p^\mu\rangle) = 0 \Rightarrow \xi_\mu \alpha^\mu_0 |p^\mu\rangle = 0 \Rightarrow \xi_\mu p^\mu |p^\mu\rangle = 0. \tag{9.38}$$

where we have used $L_1 = \Sigma_{p\geq 1} \alpha^\mu_{1-p}\alpha_{p\mu} = \alpha^\mu_0 \alpha_{1\mu} + \alpha^\mu_{-1}\alpha_{2\mu} + ...$ Thus we see that this physical string state is defined in exactly the same way as a one-photon state in QED.

What is still lacking is the effect of gauge invariance on the polarisation tensor $\xi_\mu \to \xi_\mu + i p_\mu \epsilon$, where ϵ is the Fourier transform of the gauge parameter in the Maxwell theory. The string theory origin of this gauge transformation is the existence of a null state. Consider the descendent state

$$|d\rangle := L_{-1}\frac{1}{\sqrt{2\alpha'}} i\epsilon |p^\mu\rangle = i\epsilon p_\mu \alpha^\mu_{-1}|p^\mu\rangle. \tag{9.39}$$

This state is also primary (see above) and thus a null state (for $p^2 = 0$). Thus the equivalence relation involving null states given above becomes in this case just gauge invariance

$$\xi_\mu \sim \xi_\mu + i\epsilon p_\mu. \tag{9.40}$$

So, the string implies that these states correspond to $D-2$ degrees of freedom in D spacetime dimensions exactly as we have seen is the case for a vector gauge field in field theory (e.g., in the light-cone gauge).

In the closed string the corresponding situation is realised for the states with two vector indices, i.e., the metric and the Kalb-Ramond fields.

Note that $a = -1$ is required for this analysis of the degrees of freedom to work!

The string loop expansion and the string coupling constant:
The following discussion requires some more advanced methods that we will not have time to go through in this course. So the purpose of including this brief account based on a Feynman path integral is to give a hint of what kind of mathematics is needed and how modern string theory is often formulated.

Consider the **Feynman path integral** on the world-sheet

$$Z = \int \mathcal{D}(h, X) e^{\frac{i}{\hbar} S[h,X]}. \tag{9.41}$$

This object does in principle include all information about the theory. The functional integral is over all possible field configurations on any world-sheet manifold. It may be viewed as an infinite dimensional integral over the coefficients in a general mode expansion of the functions involved.

This path integral Z can be turned into something that is a bit easier to deal with by a Wick rotation to Euclidean signature on the world-sheet

$$Z_E = \int \mathcal{D}(h, X) e^{-\frac{1}{\hbar} S_E[h,X]}. \tag{9.42}$$

We can then use the theory of *Riemann surfaces* to say that the functional integral over $h_{\alpha\beta}$ and X^μ split into separate terms for each topologically different Riemann surface. It is known that in two dimensions these surfaces are classified by one number only, the so called *genus*, denoted g. It simply corresponds to the number of holes in any two-dimensional surface without boundaries. These surfaces can be considered as multi-hole generalisations of an ordinary torus (having one hole). Thus we have

$$Z_E = \Sigma_{g=0}^{\infty} \int \mathcal{D}^{(g)}(h, X) e^{-\frac{1}{\hbar} S_E[h,X]}. \tag{9.43}$$

At this point we need to define the action $S_E[h, X]$. If the target spacetime the string is moving in is curved the metric $\eta_{\mu\nu}$ must be replaced by the corresponding curved metric $g_{\mu\nu}$. The closed string has, however, two other massless fields, the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton ϕ. If these fields are non-zero they should be included as background fields in the Polyakov type action $S_E[h, X]$. Without going into details here, we just quote the answer

$$\begin{aligned} S_E[h, X] = & -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \left(\sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) + 2\pi\alpha' \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) \right) \\ & - \frac{1}{4\pi} \int d\tau d\sigma \sqrt{-h} R_2(h) \phi(X). \end{aligned} \tag{9.44}$$

This has become an extremely complicated action functional since the background fields depend on X^μ. However, one can draw one very important conclusion by setting $\phi(X) = \langle\phi\rangle + \varphi(X)$ where the background value $\langle\phi\rangle$ is constant. Then the above path integral can be written (with $\hbar = 1$)

$$Z_E = \Sigma_{g=0}^{\infty} \, g_s^{-\chi} \int \mathcal{D}^{(g)}(h, X) e^{-S_E[h,X]}. \tag{9.45}$$

Here we have introduced a topological quantity[19], the Euler number $\chi := 2 - 2g$, and as before defined the string coupling constant by $g_s := e^{\langle\phi\rangle}$. The reason the Euler number appears here can be seen from the **Gauss-Bonnet theorem** which, when used in two dimensions, says that

$$\chi := \frac{1}{4\pi}\int d^2\sigma\sqrt{-h}R_2(h). \tag{9.46}$$

The reason behind this formula is the fact that the integrand $\sqrt{-h}R(h)$ is a total derivative in two (and only two) dimensions. This resembles what happens if we integrate the Maxwell field strength over a two-dimensional manifold which also leads to a topological result. Thus we see that $e^{\langle\phi\rangle}$ plays the role of a loop counting parameter which also explains the definition of the string coupling constant $g_s := e^{\langle\phi\rangle}$.

A last comment in this context concerns the relation to Newton's constant. From the definition $g_s := e^{\langle\phi\rangle}$ and the fact that the low energy effective action in string theory contains an Einstein-Hilbert term of the form (as indicated by the g_s factor in Z_E above)

$$S[g_{\mu\nu},\phi] = \frac{1}{(\alpha')^{\frac{D-2}{2}}}\int d^Dx\,\sqrt{-g}e^{-2\phi}R(g), \tag{9.47}$$

we see that Newton's constant normally defined as the parameter multiplying $\int d^Dx\sqrt{-g}R(g)$ is, as mentioned before, given by

$$G_N^{(D)} \sim g_s^2(\alpha')^{\frac{D-2}{2}} \sim g_s^2 l_s^{D-2}. \tag{9.48}$$

Comment: We have now obtained some understanding of α' and the string coupling constant g_s and their role as parameters of the low energy effective action in spacetime. However, there is one more important aspect related to α'. To discuss this we return to the Polyakov action for an unspecified curved spacetime metric $g_{\mu\nu}(X)$

$$S[X^\mu, h_{\alpha\beta}] = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-h}\,h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\,g_{\mu\nu}(X). \tag{9.49}$$

This action functional is extremely complicated and cannot be analysed exactly by any known method. Thus we have to resort to other methods like two-dimensional perturbation theory on the world-sheet. It does not require too much work to see that α' is the loop expansion parameter in the world-sheet perturbation theory and that $g_{\mu\nu}(X)$ plays the role of an infinite set of coupling constants (from its Taylor expansion).
One can then start computing one-loop diagrams on the world-sheet. By requiring that the conformal invariance survives at the one-loop level[20] one discovers that the metric

[19] The Euler number for any two-dimensional compact manifold can also be computed as $\chi = b_0 - b_1 + b_2$ by counting the corners (b_0), the edges (b_1) and sides (b_2). The cube gives then directly $\chi = 8 - 12 + 6 = 2$.

[20] This condition is equivalent to demanding the vanishing of all beta-functions, which are functionals of the metric and the other massless fields. This procedure is similar to the Ricci flow methods (here renormalisation group flow) used by Hamilton and Perelman to prove the Poincaré conjecture, see, e.g., Frenkel et al, hep-th/2011.11914.

(and the other massless fields) satisfy non-linear generally covariant field equations, that is, Einstein's equations etc. This way we can derive the supergravity theories that we have mentioned above should come out as low-energy approximations of string theory.

Finally, one might wonder if the Polyakov action has a role to play for higher-dimensional objects in string theory and M-theory. For instance, in M-theory we are interested in the M2-brane and it turns out to be very interesting to compare this case to the fundamental string. By a slightly more complicated calculation than for the string one can show that a three-dimensional Nambu-Goto type action for the M2-brane there is a Polyakov form reading $(\alpha, \beta, .. = 0, 1, 2)$

$$S(M2) = -\frac{T_2}{2}\int d^3\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} + \Lambda_2 \int d^3\sigma\sqrt{-h}, \;\; \Lambda_2 = \frac{T_2}{2}. \tag{9.50}$$

We note that this action has a "cosmological term" which in fact appears for all other world-sheets of dimensions $p+1$ except the $p+1 = 2$, i.e., the string. (This requires in general a cosmological term with $\Lambda_p = \frac{p-1}{2}T_p$.) We also see that it is only the string that is locally scale invariant (or conformal). Furthermore, there is no simple analogue of the conformal gauge which makes the world-sheet look flat since there is not enough local symmetries to set the metric $h_{\alpha\beta}$ equal to $\eta_{\alpha\beta}$.

It is an interesting fact in the context of the $M2$-brane that gravity in both two and three spacetime dimensions have no propagating degrees of freedom. However, while the Einstein-Hilbert term in two dimensions is a total derivative (and thus topological) the one in three dimensions is not a total derivative. Instead, in three dimensions there are Chern-Simons versions of both Yang-Mills theory and gravity which enter modern formulations on $M2$-branes in a crucial way. The Yang-Mills Chern-Simons theory is also topological[21]. This is a very active area of research.

[21] The abelian version of Chern-Simons theory in three spacetime dimensions is important in condensed matter systems where boundary degrees of freedom are studied, e.g., the fractional quantum Hall effect.

10 Lectures 10 - 12

The focus of this lecture and the next two will be on the supersymmetric string, both open and closed. We will follow closely Chapter 14 in BZ but we will add some material on supersymmetry and supergravity in a two-dimensional spacetime (the world-sheet) as a direct generalisation of the bosonic Polyakov action. We will also give examples of supergravity theories in higher-dimensional spacetimes, and some of their properties.

10.1 Chapter 14: Basic superstring theory

The reason for not being too pleased about the bosonic string is twofold:

1) Bosonic string theory has a tachyon and is therefore unstable.

2) The spectrum of the bosonic string does not contain any spacetime fermions.

To solve the second problem the natural assumption is that one has to introduce fermionic fields already on the world-sheet. This turns out to be true (almost due to bosonisation[22]) but the surprise is that it also solves the first problem as we will see later.

Another very important point (maybe not emphasised enough in textbooks) is that if we introduce fermions on the world-sheet we must do it in a supersymmetric way. So, although nature is not supersymmetric (we have not seen any experimental evidence for broken spacetime supersymmetry yet) we cannot avoid it when constructing a string incorporating spacetime fermions. This will be clear below.

Question: What kind of fermionic fields should we use together with X^μ to construct a supersymmetric Polyakov type field theory on the world-sheet?

Answer: There are in fact (at least) three rather different ways to do this[23]:

1) $\psi_a^\mu(\tau, \sigma)$, where a is a world-sheet spinor index (taking two values). This leads to the so called NSR formalism[24] and supersymmetry on the world-sheet (discussed in detail below).

2) $\Theta^A(\tau, \sigma)$ is a scalar on the world-sheet. The index A is a spinor index in the target spacetime (with $D = 10$) taking 16 values. This is called the GS formalism (for Green-Schwarz) and leads naturally to supersymmetry in spacetime instead of on the world-sheet[25]. Not discussed further here.

[22]See the vertex operator discussion in a later lecture.
[23]It is difficult but possible to show that the NSR and the GS formulations below are equivalent.
[24]NSR refers to the names Neveu-Schwarz-Ramond.
[25]The pair X^μ, Θ^A is closely related to the superspace approach of $D = 10$ supergravity.

3) Pure spinors (developed by Berkovits): Complicated but is perhaps more fundamental than the other two formulations. Not discussed further here.

NSR: Superstring theory based on $(X^\mu(\tau,\sigma), \psi^\mu_a(\tau,\sigma))$ will necessarily be a supergravity theory on the world-sheet, i.e., it must possess local supersymmetry. The reason for this is the same as for the reparametrisation invariance of the bosonic Polyakov theory, namely that it must be possible to use some gauge invariance to eliminate the time component of X^μ. For the superstring also the time component of ψ^μ_a must be removable which, however, requires a local symmetry with anti-commuting parameters, i.e., **world-sheet supergravity**. To describe how supersymmetry arises in target space is more complicated but will be carefully explained below. If you are unfamiliar with the concept of supersymmetry you will get an introduction as we go along.

To simplify this rather complicated story a bit we will here develop the theory in two steps: (only the first step appears in BZ, but for transverse directions only):

1) Using a Polyakov type action for $X^\mu(\tau,\sigma), \psi^\mu_a(\tau,\sigma)$ on a **flat world-sheet global supersymmetry** can be demonstrated. We can also find the field equations and boundary conditions without too much work (this lecture).

2) To get the super-generalisation of the bosonic stress tensor and constraints we need to discuss the **supergravity version of the Polyakov action**. Without giving all the details we will sketch how the supergravity action gives rise the superconstraints and thus to the super-Virasoro algebra (the coming two lectures).

Global supersymmetry:
The Polyakov action for the string coordinates $X^\mu(\tau,\sigma), \psi^\mu_a(\tau,\sigma)$ on a flat world-sheet is given by

$$S[X^\mu, \psi^\mu_a] = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \left(\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} + \frac{i}{2}\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu}\right), \tag{10.1}$$

where $\bar{\psi}^\mu$ is the Dirac conjugate and the two-dimensional Dirac matrices used here are (in terms of Pauli matrices)

$$\rho^\alpha = (\rho^0, \rho^1) = (\epsilon, -\sigma^1), \text{ where } \epsilon = i\sigma^2. \tag{10.2}$$

They satisfy $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$. We will also need $\rho^3 := \rho^0\rho^1 = -\sigma^3$.

As for the bosonic part of the action there is a factor of $\frac{1}{2}$ in the Dirac term due to the fact that the spinor field ψ^μ_a is Majorana and hence real (as are X^μ): Recall the Majorana condition $\bar{\psi} = \psi^T C$. Here the charge conjugation matrix on the world-sheet satisfies the standard relations $C\rho^\alpha C^{-1} = -(\rho^\alpha)^T$. Thus we can set $C = \rho^0$ which implies $\psi^\dagger = \psi^T$. So ψ is a real two-component spinor for each value of μ. Note that also the Dirac matrices are real.

To show global supersymmetry and to find the field equations and the possible boundary conditions we first perform a general variation of the fields in the action. We get

$$\delta S[X^\mu, \psi_a^\mu] = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \left(\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta \delta X^\nu \eta_{\mu\nu} + \frac{i}{2}\bar{\psi}^\mu \rho^\alpha \partial_\alpha \delta\psi^\nu \eta_{\mu\nu} + \frac{i}{2}\delta\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu}\right). \tag{10.3}$$

Integrating the first and second terms by part (to isolate the variations) this becomes

$$\begin{aligned}\delta S[X^\mu, \psi_a^\mu] &= -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \left(-(\Box_2 X^\mu)\delta X^\nu \eta_{\mu\nu} - \frac{i}{2}(\partial_\alpha \bar{\psi}^\mu \rho^\alpha)\delta\psi^\nu \eta_{\mu\nu} + \frac{i}{2}\delta\bar{\psi}^\mu(\rho^\alpha \partial_\alpha \psi^\nu)\eta_{\mu\nu}\right) \\ &- \frac{1}{2\pi\alpha'}\int d\tau d\sigma \left(\partial_\beta(\eta^{\alpha\beta}\partial_\alpha X^\mu \delta X^\nu \eta_{\mu\nu}) + \frac{i}{2}\partial_\alpha(\bar{\psi}^\mu \rho^\alpha \delta\psi^\nu \eta_{\mu\nu})\right). \end{aligned} \tag{10.4}$$

Our first task is to check global supersymmetry. Dropping the boundary terms and using the **supersymmetry transformations**

$$\delta_\epsilon X^\mu = i\bar{\epsilon}\psi^\mu, \;\; \delta_\epsilon \psi^\mu = \rho^\alpha \epsilon \partial_\alpha X^\mu, \;\; \delta_\epsilon \bar{\psi}^\mu = -\bar{\epsilon}\rho^\alpha \partial_\alpha X^\mu, \tag{10.5}$$

where ϵ is a constant world-sheet **anti-commuting** Majorana spinor, we find

$$\delta_\epsilon S[X^\mu, \psi_a^\mu] = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \left(-\Box_2 X^\mu i\bar{\epsilon}\psi^\nu \eta_{\mu\nu} - \frac{i}{2}(\partial_\alpha \bar{\psi}^\mu \rho^\alpha)\rho^\beta \epsilon \partial_\beta X^\nu \eta_{\mu\nu} - \frac{i}{2}\bar{\epsilon}\rho^\alpha \partial_\alpha X^\mu(\rho^\beta \partial_\beta \psi^\nu)\eta_{\mu\nu}\right). \tag{10.6}$$

To see that these terms cancel we integrate by parts again (dropping boundary terms) to get all terms to contain $\Box_2 X^\mu$. Thus, using $\rho^\alpha \rho^\beta \partial_\alpha \partial_\beta = \Box_2$,

$$\delta_\epsilon S[X^\mu, \psi_a^\mu] = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \,(-i(\Box_2 X^\mu)\eta_{\mu\nu})\,(\bar{\epsilon}\psi^\nu - \frac{1}{2}\bar{\epsilon}\psi^\nu - \frac{1}{2}\bar{\psi}^\nu \epsilon) = 0. \tag{10.7}$$

To see that $\delta_\epsilon S = 0$, we note that since the spinors are Majorana we have $\bar{\psi}\epsilon = \psi^T C\epsilon = -\epsilon^T C^T \psi = \bar{\epsilon}\psi$ and the last two terms cancel the first term.

We now derive the field equations and possible boundary conditions, which are very important in the fermionic sector. Returning to the general variation of the action above, Hamilton's principle says that we should set $\delta S = 0$ and consider the bulk and boundary terms separately. The vanishing of the bulk terms gives immediately the field equations

$$\Box_2 X^\mu = 0, \;\; \rho^\alpha \partial_\alpha \psi^\mu = 0. \tag{10.8}$$

The boundary terms need some rewriting before becoming useful. The bosonic sector is rather simple: We have, respectively for the open and closed string,

$$X'^\mu \delta X^\nu \eta_{\mu\nu}|_{\sigma=0}^{\sigma=\pi} = 0, \;\; X'^\mu \delta X^\nu \eta_{\mu\nu}|_{\sigma=0} = X'^\mu \delta X^\nu \eta_{\mu\nu}|_{\sigma=2\pi}. \tag{10.9}$$

As already discussed, these open string boundary conditions can be satisfied by either imposing Dirichlet, $\delta X^\mu = 0$, or Neumann, $X'^\mu = 0$, boundary conditions independently at the two ends and for each μ-component. The closed string conditions are satisfied by imposing periodic boundary conditions $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)$. Note that the boundary terms in the τ direction vanish by definition.

Turning to the fermionic boundary conditions the situation changes completely. Here we should first check what the field equations tell us. They become, if written in components, and with the above ρ^α matrices, $\sigma^\pm := \tau \pm \sigma$ and $\partial_\pm := \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$,

$$\rho^\alpha \partial_\alpha \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} = \begin{pmatrix} 0 & \partial_0 - \partial_1 \\ -(\partial_0 + \partial_1) & 0 \end{pmatrix} \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} = 0 \Rightarrow \partial_+ \psi^\mu_- = 0, \;\; \partial_- \psi^\mu_+ = 0. \tag{10.10}$$

This implies that $\psi^\mu_+(\tau,\sigma) = \psi^\mu_+(\sigma^+)$ and $\psi^\mu_-(\tau,\sigma) = \psi^\mu_-(\sigma^-)$. Note that the solution to the Klein-Gordon equation contains two similar functions since $X^\mu(\tau,\sigma) = f^\mu(\sigma^+) + g^\mu(\sigma^-)$.

The boundary conditions (in the σ direction) obtained above read

$$\eta_{\mu\nu}(\bar\psi^\mu \rho^1 \delta\psi^\nu)| = 0. \tag{10.11}$$

Written out in components they become, using $\bar\psi\rho^1\psi = \psi^T C \rho^1 \psi = \psi^T \rho^0 \rho^1 \psi = \psi^T \rho^3 \psi$,

$$\eta_{\mu\nu}(\psi^\mu_- \delta\psi^\nu_- - \psi^\mu_+ \delta\psi^\nu_+)| = 0. \tag{10.12}$$

For the open string this condition must be satisfied independently for the two end points. The relative minus sign now provides a new possibility, namely to relate the two components of ψ^μ. So the boundary conditions are satisfied at both ends if we set

$$\psi_+(\tau,0) = \psi_-(\tau,0), \;\;\; \psi_+(\tau,\pi) = \pm\psi_-(\tau,\pi) \text{ where } \begin{cases} \text{plus sign: Ramond (R) sector} \\ \text{minus sign: Neveu-Schwarz (NS) sector.} \end{cases} \tag{10.13}$$

Choosing a minus sign in the $\sigma = 0$ relation turns out to give nothing new so this case will not be used in the following. In a manner similar to the bosonic string we can combine the two $\psi_\pm$ components into one fermionic (anticommuting) function with 2π boundary conditions as follows

$$\Psi^\mu(\tau,\sigma) := \begin{cases} \psi_-(\tau,\sigma) = \psi_-(\sigma^-), \text{ for } \sigma \in [0,\pi], \\ \psi_+(\tau,-\sigma) = \psi_+(\sigma^-), \text{ for } \sigma \in [-\pi,0]. \end{cases} \tag{10.14}$$

Then the R (NS) boundary condition become 2π (anti)periodic ones on Ψ^μ:

$$\Psi^\mu(\tau,\pi) = \psi^\mu_-(\tau,\pi) = \pm\psi^\mu_+(\tau,\pi) = \pm\Psi^\mu(\tau,-\pi). \tag{10.15}$$

Thus we see that the R sector is the one with periodic boundary conditions and the NS sector the one with anti-periodic boundary conditions. This fact has a tremendous impact on the mode expansions:

$$\text{R sector:} \quad \Psi^\mu(\tau,\sigma) = \Sigma_{n\in\mathbf{Z}}\, d^\mu_n\, e^{-in(\tau-\sigma)}, \tag{10.16}$$

$$\text{NS sector:} \quad \Psi^\mu(\tau,\sigma) = \Sigma_{r\in\mathbf{Z}+\frac{1}{2}}\, b^\mu_r\, e^{-ir(\tau-\sigma)}, \tag{10.17}$$

Note the half-integer sum in the NS case. Notation: we will always let m, n, p, q represent integers and $r, s, ..$ half-integers.

Without going through the canonical quantisation in detail for the anti-commuting fermionic fields, we just quote the result:

$$\text{R sector:} \quad \{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0}, \tag{10.18}$$

$$\text{NS sector:} \quad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0}. \tag{10.19}$$

We will now analyse the spectrum of the superstring. There are several tricky points involved so we will do this in the light-cone formalism to avoid some of them. After this is done we return to the covariant formalism and the world-sheet supergravity theory to get a feeling for the super-constraints and the super-Virasoro algebra.

The superstring spectrum in the light-cone formalism:
The condition that the spacetime Lorentz commutator $[M^{-I}, M^{-J}] = 0$ in the superstring will tell us that $D = 10$ is the critical dimension where the superstring lives. This will not be shown here so we will just assume it to be true. We first consider the NS sector since this is the easier one.

The NS spectrum:
The modes in this sector are b^I_r for all $r \in \mathbf{Z} + \frac{1}{2}$. We define the ones with a negative r to be creation operators, i.e., for $r > 0$ we have $b^I_{-r} := b^{I\dagger}_r$ (since ψ^I are Majorana). Thus, we can introduce a ground state $|p^+, p^I\rangle_{(NS)}$, which has a factor $|0\rangle_r$ for each positive value of r in it, satisfying

$$b^I_r|p^+, p^I\rangle_{(NS)} = 0, \quad r = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, ... \tag{10.20}$$

The space of states is then generated by acting with all possible combinations of b^I_{-r} for $r > 0$ (remembering that they anticommute) and α^I_{-n}. Thus the space of states in the NS sector is given by

$$|\lambda\rangle_{(NS)} = \Pi^{I=9}_{I=2}\,\Pi^\infty_{n=1}(\alpha^I_{-n})^{\lambda^{NS}_{I,n}}\,\Pi^{J=9}_{J=2}\,\Pi^\infty_{r=\frac{1}{2}}(b^J_{-r})^{\tilde\lambda^{NS}_{J,r}}\,|p^+, p^I\rangle_{(NS)}, \tag{10.21}$$

where $\lambda^{NS}_{I,n}$ can be any non-negative integer and $\tilde\lambda^{NS}_{J,r}$ zero or one. This expression is sometimes written in a bit more compact way as $\alpha^{\{I\}}_{\{-n\}} b^{\{J\}}_{\{-r\}}|p^+, p^I\rangle_{(NS)}$.

Note 1: There are no zero modes in ψ^μ in the NS sector!
Note 2: At the three lowest levels the states are $|p^+, p^I\rangle_{(NS)}$, $b^I_{-\frac{1}{2}}|p^+, p^I\rangle_{(NS)}$, $\alpha^I_{-1}|p^+, p^I\rangle_{(NS)}$ and $b^I_{-\frac{1}{2}} b^J_{-\frac{1}{2}}|p^+, p^I\rangle_{(NS)}$. This fact will create a problem with spin-statistic that we must, and will, resolve later.

As will also become clear later the mass operator in the superstring case contains number operators from both the bosonic (α_n) sector and either the NS or R fermionic sector. In the open string NS sector we hence have

$$M^2 = \frac{1}{\alpha'}(\frac{1}{2}\Sigma_{n>0}\, \alpha^I_{-n}\alpha^I_n + \frac{1}{2}\Sigma_{r>0}\, r\, b^I_{-r}b^I_r - \frac{1}{2}). \tag{10.22}$$

Here we should note that a factor of r appears in the second sum. This is related to the fact that $\{b_r, b^\dagger_{-s}\} = \delta_{rs}$ (for each r) and not $= r\delta_{rs}$.

The $-\frac{1}{2}$ in M^2 is the normal ordering constant (in $D = 10$) now having contributions from both the α_n and the b_r terms: $a_{NS} := a_\alpha + a_b$ with

$$a_\alpha = \frac{D-2}{2}\Sigma^\infty_{n=1}\, n = -\frac{D-2}{24} = -\frac{1}{3}, \tag{10.23}$$

$$a_b = -\frac{D-2}{2}\Sigma^\infty_{r=\frac{1}{2}}\, r = -\frac{D-2}{48} = -\frac{1}{6}, \tag{10.24}$$

where we have used $D = 10$ and that the b operators anticommute. The sum over half-integers was derived in a previous lecture.

To summarise, for the open superstring the mass spectrum in the NS sector is given by

$$M^2 = \frac{1}{\alpha'}(N^\perp_\alpha + N^\perp_b - \frac{1}{2}). \tag{10.25}$$

This gives the following low level states and mass values (defining $|NS\rangle := |p^+, p^I\rangle_{(NS)}$)

$$|NS\rangle : \quad \alpha' M^2 = -\frac{1}{2} \text{ (tachyon)} \tag{10.26}$$

$$b^I_{-\frac{1}{2}}|NS\rangle : \quad \alpha' M^2 = 0 \text{ (massless vector field)} \tag{10.27}$$

$$\alpha^I_{-1}|NS\rangle,\ b^I_{-\frac{1}{2}}b^J_{-\frac{1}{2}}|NS\rangle : \quad \alpha' M^2 = \tfrac{1}{2} \text{ (massive vector field)}. \tag{10.28}$$

We see that the tachyon problem is back and that we clearly have a problem with spin-statistics! If we choose the ground-state to be anti-commuting then all states (continue the above list to higher level) with integer M^2 are commuting (bosonic) and those with half-integer M^2 are all anti-commuting (fermionic). Since also these latter states are tensors (not spinors) they do not respect the spin-statistics theorem. Before dealing with these problems we must understand the R sector which has some interesting new features. Another problem is that we still have not seen any sign of spinors representations in spacetime.

The R spectrum:
The modes in this sector are d^I_n (for all $n \in \mathbf{Z}$) satisfying the CCR

$$\{d^I_m, d^J_n\} = \delta^{IJ}\delta_{m+n,0}. \tag{10.29}$$

We define the ones with a negative n to be creation operators, i.e., for $n > 0$ we define $d^I_{-n} := d^{I\dagger}_n$. Thus, we can introduce a ground state $|p^+, p^I\rangle_{(R)}$, which has a factor $|0\rangle_n$ for each $n > 0$, satisfying

$$d^I_n|p^+, p^I\rangle_{(R)} = 0, \quad n = 1, 2, ... \tag{10.30}$$

The space of states is then generated by acting with all possible combinations of $d^I_{-n} := (d^I_n)^\dagger$ for $n > 0$ (remembering that they anticommute). Thus the space of states in the R sector is given by

$$|\lambda\rangle_R = \Pi_{I=2}^{I=9}\Pi_{n=1}^{\infty}(\alpha^I_{-n})^{\lambda^R_{I,n}}\Pi_{J=2}^{J=9}\Pi_{m=1}^{\infty}(d^J_{-m})^{\tilde{\lambda}^R_{J,m}}|p^+, p^I\rangle_{(R)}, \tag{10.31}$$

where $\lambda^R_{I,n}$ can be any non-negative integer but $\tilde{\lambda}^R_{J,m}$ only zero or one. This expression is sometimes written more compactly as $\alpha^{\{I\}}_{\{-n\}}d^{\{J\}}_{\{-m\}}|p^+, p^I\rangle_{(R)}$.

However, in the R sector this is not the full story since there is also a set of zero modes d^I_0. These have to be treated with care since they do not seem to follow the general rule that there should be a natural pair of annihilation and creation operators. How do we handle this situation?

The key observation is that the CCR for the zero modes is

$$\{d^I_0, d^J_0\} = \delta^{IJ}, \tag{10.32}$$

which therefore can be represented by $SO(8)$ gamma matrices by $d^I_0 := \frac{1}{\sqrt{2}}\gamma^I$. Since Ψ^I is a real field d^I_0 are also real, or hermitian after quantisation. So the question is then: How do we construct the space of states generated by d^I_0?

If one is familiar with Dirac matrices (e.g., from QFT) the answer is immediately clear: The space of states transform as spinor representations of $SO(8)$. To see this explicitly one can construct the space of states and then check the statement that these states together define a spinor. This is done as follows: Construct four pairs of creation and annihilation operators from the eight hermitian operators d^I_0 by

$$\xi_i := \frac{1}{\sqrt{2}}(d^{2i-1}_0 + id^{2i}_0), \;\; \xi^\dagger_i := \frac{1}{\sqrt{2}}(d^{2i-1}_0 - id^{2i}_0), \;\; i = 1, 2, 3, 4. \tag{10.33}$$

These satisfy

$$\{\xi_i, \xi^\dagger_j\} = \delta_{ij}. \tag{10.34}$$

Define now a ground state as usual by $\xi_i|0\rangle = 0$. This means that the space of states in the zero mode sector of the R sector contains the states

$$|0\rangle, \; \xi^\dagger_i|0\rangle := |i\rangle, \; \xi^\dagger_i\xi^\dagger_j|0\rangle := |[ij]\rangle, \;\; \xi^\dagger_i\xi^\dagger_j\xi^\dagger_k|0\rangle := |[ijk]\rangle, \;\; \xi^\dagger_i\xi^\dagger_j\xi^\dagger_k\xi^\dagger_l|0\rangle := |[ijkl]\rangle. \tag{10.35}$$

These $2^4 = 16 = 1 + 4 + 6 + 4 + 1$ states transform into each other under the 28 operators $d^I_0 d^J_0$ (since these are just a sum of terms bilinear in the ξ_is and $\xi^\dagger_i$s). In fact, one can check (or just compare to the matrices γ^{IJ}) that these 28 operators generate the $SO(8)$

Lie algebra. From this we conclude that the 16 states is a spinor under $SO(8)$.

Now that we have understood the state space in the R zero mode sector, there is one more very important point that we will need later. Looking at the zero mode space of states above we note that half of them have an even number of creation operators in them and half of them an odd number. But the operators $d_0^I d_0^J$ are bilinear in these ξ operators so they do not mix these two sets of states. As representations of the Lie algebra of $SO(8)$ these states therefore make up two chiral, left and right, spinorial irreps (=irreducible representations). The possibility to split the 16 states into these two sets of chiral spinor irreps will be of paramount importance below. We will denote these two 8-component chiral spinors as, with $|0\rangle_{(R)}$ (adding the (R) for clarity) being a component of $|R_a\rangle$,

$$\text{left}: \quad |R_a\rangle, \quad \text{right:} \ |R_{\bar{a}}\rangle. \tag{10.36}$$

Again there is clearly a spin-statistics problem: If $|0\rangle_{(R)}$ is chosen to be odd, (that is, anticommuting[26] since it is a state in a spinor which is an anti-commuting object) then the states of the two spinors $|R_a\rangle$, $|R_{\bar{a}}\rangle$ will have opposite statistics properties even though they are both spinors in spacetime. Thus one of them (here $|R_{\bar{a}}\rangle$) will necessarily violate the spin-statistics theorem.

As we will now explain, however, the spin-statistics problems in both the NS and R sectors, together with the tachyon problem in the NS sector, will be solved in one blow by the so called GSO projection. To understand how this is done we need to study the R sector a bit more.

The mass spectrum in the R sector is determined by

$$M^2 = \frac{1}{\alpha'}\Sigma_{n\geq 1}\left(\alpha^I_{-n}\alpha^I_n + n d^I_{-n} d^I_n\right). \tag{10.37}$$

Here we note that the presence of the factor n in the second term implies that the zero modes d_0^I do not contribute to the mass operator. In other words, all the spinorial states constructed from only d_0^I operators, or rather the $\xi_i^\dagger$, are degenerate in mass, i.e., they have all zero mass. This is of course a key property if we want all spinor states to correspond to Dirac fields with a given mass in spacetime. Remember also that in the Standard Model all fermions are massless before the Higgs effect sets in.

Note also the curious fact that the normal ordering constant is zero in the R sector. This is a direct consequence of the fact that the infinite sums are the same for the α and the d terms in M^2 and that the d operators anti-commute.

[26]"Odd" means "odd Grassmann". "Even" and "odd" are often used when referring to objects that are either commuting, i.e., even Grassmann, or anti-commuting, i.e., odd Grassmann.

By assuming that the state $|0\rangle_{(R)}$ is fermionic (anti-commutiing), the low level part of the R sector spectrum may be listed as:

$$\text{Fermionic states: } |R_a\rangle, \alpha^I_{-1}|R_a\rangle,\ d^I_{-1}|R_{\bar{a}}\rangle, ... \tag{10.38}$$

$$\text{Bosonic states: } |R_{\bar{a}}\rangle, \alpha^I_{-1}|R_{\bar{a}}\rangle,\ d^I_{-1}|R_a\rangle, ... \tag{10.39}$$

Recall that the state $|0\rangle_{(R)}$ belongs to the left-chiral spinor denoted $|R_a\rangle$ and note that the chirality of the tensor-spinor irreps shifts from state to state in each list if they differ by a odd number of d^I_{-n} operators. This follows by checking the chirality by applying the operators $P_{L/R} = \frac{1}{2}(1 \pm \gamma^9)$ to the state and observe that $|R_a\rangle$ and $\gamma^I|R_a\rangle$ have opposite chirality properties since $\gamma^9\gamma^I = -\gamma^I\gamma^9$. As usual $\gamma^9 := \gamma^1...\gamma^8$.

State counting and the *GSO* projection to the physical theory:
It is a nice mathematical feature of string theory that it is possible to "count" all the states in the infinite spectrum and describe the result in terms of "spectrum generating functions" of one variable $f(x)$[27].

As a concrete and simple example consider one pair of creation-annihilation operators $(a, a^\dagger)$ satisfying $[a, a^\dagger] = 1$. The state space generated by powers of $a^\dagger$ acting on the ground state $|0\rangle$ is

$$|0\rangle,\ a^\dagger|0\rangle,\ (a^\dagger)^2|0\rangle,\ (a^\dagger)^3|0\rangle, ... \tag{10.40}$$

The number of states at each level n, the eigenvalue of the number operator $N = a^\dagger a$, is denoted A_n and is here just

$$A_0 = 1,\ \ A_1 = 1,\ \ A_2 = 1, ... \tag{10.41}$$

This spectrum can thus be represented by a "spectrum generating function" constructed from the values of A_n by

$$f_1(x) := \Sigma^\infty_{n=0} A_n x^n = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}. \tag{10.42}$$

Now consider two pairs of such operators $a_i, a^\dagger_i$, $i = 1, 2$ with $[a_i, a^\dagger_j] = \delta_{ij}$. Then the state space is

$$|0\rangle,\ a^\dagger_i|0\rangle,\ a^\dagger_i a^\dagger_j|0\rangle,\ a^\dagger_i a^\dagger_j a^\dagger_k|0\rangle, ... \tag{10.43}$$

The degeneracy now becomes

$$\tilde{A}_0 = 1,\ \ \tilde{A}_1 = 2,\ \ \tilde{A}_2 = 3,\ \ \tilde{A}_3 = 4,\ \ \tilde{A}_4 = 5.... \tag{10.44}$$

generated by

$$\tilde{f}_1(x) := \Sigma^\infty_{n=0} \tilde{A}_n x^n = 1 + 2x + 3x^2 + 4x^3 + ... = (\frac{1}{1-x})^2 = (f_1(x))^2. \tag{10.45}$$

[27]This technique is of enormous importance in string theory in the one-loop context, where these functions become *modular forms*.

The generalisation to several such pairs of operators will be used below.

Another kind of generalisation that appears in string theory is to $a_2, a_2^\dagger$ also satisfying $[a_2, a_2^\dagger] = 1$ but now with a number operator $N_2 := 2a_2^\dagger a_2$. Then the state space is the same as above for one pair but the eigenvalues of N_2 are $0, 2, 4, ...$ which leads to the generating function

$$f_2(x) = 1 + x^2 + x^4 + ... = \frac{1}{1-x^2}. \tag{10.46}$$

Having two such pairs leads to a function that is the square of $f_2(x)$ just as in the above example.

The bosonic string "mass level generating function" is then

$$f_{bose}(x) = \frac{1}{x}\Pi_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)^{24}, \tag{10.47}$$

obtained using $(a_n^I, a_n^{I\dagger})$, $N = \Sigma_{n=1}^{\infty} n a_n^{I\dagger} a_n^I$, and $\alpha' M^2 = N - 1$. The normal ordering constant -1 is the reason for the extra factor $1/x$ in $f_{bose}(x)$.

The reason for constructing these functions from the mass spectrum is that in a spacetime supersymmetric theory the number of bosonic dof in the spacetime low energy field theory equals the number of fermionic dof (not proven in this course). So, by constructing the generating functions in the NS and R sectors and compare them we may get a hint how to get consistent supergravity theories in spacetime from the superstring (i.e., without a tachyon and satisfying the spin-statistics theorem).

To obtain these functions for the superstring we note that for anti-commuting canonical operators like $(b_r^I, b_r^{I\dagger})$ and $(d_n^I, d_n^{I\dagger})$ each creation operator can only occur once in the state. For $(b_r^I, b_r^{I\dagger})$ this means simply that $f_r^{NS} = (1 + x^r)^8$. The whole NS sector then gives the generating function

$$f^{NS}(x) = \frac{1}{\sqrt{x}}\Pi_{r=\frac{1}{2}}^{\infty}(1+x^r)^8\Pi_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)^8 = \frac{1}{\sqrt{x}}\Pi_{n=1}^{\infty}\left(\frac{1+x^{n-\frac{1}{2}}}{1-x^n}\right)^8. \tag{10.48}$$

Similarly in the R sector we get, noting the factor 16 from the mass degenerate zero modes d_0 ,

$$f^R(x) = 16\,\Pi_{n=1}^{\infty}\left(\frac{1+x^n}{1-x^n}\right)^8. \tag{10.49}$$

Obviously these two functions are not equal, i.e., $f^{NS}(x) \neq f^R(x)$ (just check the first term) which they need to be for the spacetime spectrum to be supersymmetry as mentioned above. However, there exists a truly amazing formula proved in 1829 by Jacobi:

$$\frac{1}{2}\frac{1}{\sqrt{x}}\left(\Pi_{n=1}^{\infty}(\frac{1+x^{n-\frac{1}{2}}}{1-x^n})^8 - \Pi_{n=1}^{\infty}(\frac{1-x^{n-\frac{1}{2}}}{1-x^n})^8\right) = 8\,\Pi_{n=1}^{\infty}\left(\frac{1+x^n}{1-x^n}\right)^8. \tag{10.50}$$

Of course, using a computer one can expand this equation to any power in x and verify its validity. In string theory, this identity tells us that we should cut both the NS sector and the R sector in half to have a chance of constructing a spacetime supergravity theory. Such supergravities can be, and have been, constructed independently of string theory (by imposing supersymmetry on the Einstein-Hilbert action), and turn out to provide the correct interpretation of Jacobi's identity above. The key is the so called GSO projection introduced by Gliozzi, Olive and Scherk in a famous paper from 1977.

This GSO-projection can be shown to work also in the interacting string theory not just on the spectrum as we do here. The R sector is cut in half by the GSO projection by keeping only the chiral ground state $|R_a\rangle$, defined to be fermionic, and all the states above it which are also fermionic (see the list above). This leads to the RHS of Jacobi's identity. In the NS sector something similar is done which must include the elimination of the tachyon. Thus in the NS sector we keep all states with integer M^2. Note that all the R sector states have integer M^2 which must therefore be true in both sectors due to supersymmetry in the target spacetime. This leads to the LHS of the identity. In fact, the combination of the two terms on the LHS counts only terms, at each mass level, that have integer M^2 and are bosonic given that the ground state $|0\rangle_{(NS)}$ is anti-commuting.

Thus the GSO projection also solves the spin-statistics problem we mentioned above. So, the NS sector gives rise to all the bosonic integer spin fields in the low-energy action of the superstring and the R sector all the fermionic half-integer fields.

We have been a bit sloppy above referring to supergravity although the discussion was carried out for the open string which only contains Maxwell type gauge fields in the massless sector. We must therefore take these results over to the closed superstring which has a left-moving and a right-moving sector. The corresponding results for the open superstring is the supersymmetric Maxwell or Yang-Mills theory which we will have reason to discuss more later in the context of D-branes.

To summarise the open superstring spectrum in $D = 10$ we give the first two levels, the massless one and the lowest massive one based on anti-commuting ground states $|p^+, p^I\rangle_{(NS)}$ and $|0; p^+, p^I\rangle_{(R)}$: (the $+$ and $-$ notation below is the conventional one)

Massless open superstring states:

$$\text{NS+: } b^I_{-\frac{1}{2}}|p^+, p^I\rangle_{(NS)}, \quad \text{R-: } \; |R_a; p^+, p^I\rangle_{(R)}. \tag{10.51}$$

These states make up a target space covariant supersymmetric vector multiplet consisting of a Maxwell vector field $A_\mu(x)$ (the NS states) and an 16-component chiral Majorana spinor field $\psi(x)$ (the R states)[28].

[28]Note that in $D = 10$ Minkowski space it is possible to impose the Majorana and a chirality constraint at the same time, contrary to the case in 3+1 dimensions. Note also that although the number of covariant spinor components is 16 the number of d.o.f. is only 8.

Lowest level massive open superstring states:

$$\text{NS+: } b^I_{-\frac{1}{2}} b^J_{-\frac{1}{2}} b^K_{-\frac{1}{2}} |p^+, p^I\rangle_{(NS)} \quad \text{and} \quad b^I_{-\frac{3}{2}} |p^+, p^I\rangle_{(NS)} \qquad \text{and} \quad \alpha^I_{-1} b^J_{-\frac{1}{2}} |p^+, p^I\rangle_{(NS)}, \tag{10.52}$$

$$\text{R-: } \alpha^I_{-1} |R_a; p^+, p^I\rangle_{(R)} \quad \text{and} \quad d^I_{-1} |R_{\bar{a}}; p^+, p^I\rangle_{(R)}. \tag{10.53}$$

The bosonic NS states are $\frac{8 \cdot 7 \cdot 6}{3!} + 8 + 8 \cdot 8 = 128$ in number which is also the case for the $8 \cdot 8 + 8 \cdot 8 = 128$ states in the $R-$ sector. Note that we could equally well have chosen the $R+$ sector above since this is just a matter of picking the left or right chirality of the spinors (but it would mean using a commuting ground state $|0\rangle_{(R)}$ instead). This possibility to choose chirality in the R sector will be very important in the case of the closed superstring below.

10.2 BZ Chapter 14 cont.: More on superstrings plus extra material on supergravity and the M-theory amoeba diagram

The closed superstring:
The implications for the closed superstring of the open string discussion above are profound. In the open string case we ended up with two sectors, R and NS, and discovered the need for the GSO projection to get a spacetime supersymmetric theory. The closed superstring has a left-moving and a right-moving sector with independent sets of mode expansions and the possibility to choose their boundary conditions independently as either NS or R. This gives rise to four combinations of sectors:

$$\text{(left, right)}: \ (NS+, NS+), \ (NS+, R-), \ (R\pm, NS+), \ (R\pm, R-). \tag{10.54}$$

Several comments are needed here. The first important point is that the four spectra above are added together when defining the corresponding closed string. Also

1) $NS+$ refers to the GSO-half of the NS spectrum that does not contain the tachyon. Hence $(NS+, NS+)$ contains only bosonic fields including the massless ones $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ. These fields are part of all ordinary closed strings. Their respective number of d.o.f. is obtained in the light-cone as $\frac{1}{2}(D-2)(D-1)-1 = 35$, $\frac{1}{2}(D-2)(D-3) = 28$ and 1, together 64 bosonic d.o.f. These bosonic states are

$$|IJ; p^+, p^I\rangle_{(NS)} := b^I_{-\frac{1}{2}} \bar{b}^J_{-\frac{1}{2}} |p^+, p^I\rangle_{(NS)}, \tag{10.55}$$

which thus means that the ground state $|p^+, p^I\rangle_{(NS)}$ is defined to be commuting. Recall from the bosonic string discussion that a closed string ground state is a (tensor)product of the ground states in the left and right sectors (except for the zero modes which are common to the left and right movers).

2) $R\pm$ refers to the two GSO projected halves of the full R spectrum that contain either $|R_a\rangle$ or $|R_{\bar{a}}\rangle$, for $-$ and $+$, respectively. Using $R+$ instead of $R-$ in the second sector (and the fourth sector) in the above list of $(left, right)$ sectors does not produce a fundamentally different theory so this case is normally not discussed.

3) The spectra of the second and third sectors in the list above contain fields with one index from each of the left and right movers so they are vector-spinors in the massless part of the spectrum. These are called Rarita-Schwinger fields and are spin 3/2 anti-commuting gauge fields for the local supersymmetry that is needed to construct these supergravity theories. We will see an example of this on the world-sheet when writing down the super-Polyakov action for the below. These two Rarita-Schwinger fields correspond to two supersymmetries, usually referred to as $\mathcal{N} = 2$ theories. Each of the two fields has 64 d.o.f., and thus together 128 d.o.f..

4) There are then two physically different theories that can be defined: Either the two

Rarita-Schwinger fields have the same chirality or they have opposite chirality: The former case is called **type IIB** and the latter case **type IIA**.

5) The two theories **type IIA** and **type IIB** are physically *very different*: IIA is non-chiral and is directly related to M-theory in eleven dimensions, while IIB is chiral and has only an indirect relation to M-theory (via a T-duality transformation discussed in a later lecture).

6) The difference between IIA and IIB has important implications for the last term in the list above: $(R_{\pm}, R_{-})$ as we will see below. This issue will be analysed in the light-cone gauge. We are seeking another 64 bosonic dof to make the theory complete (i.e., supersymmetric in spacetime).

The RR sectors:
Type IIA: In this case the spectrum is $(R+, R-)$ and hence has one $SO(8)$ Weyl spinor of each chirality. Supersymmetry is then referred to as being $(1,1)$. The spin $3/2$ sectors form together a non-chiral Majorana spinor with 16 components a fact that can be used for the Rarita-Schwinger fields. In the spectrum of the $(left, right)$ sector $(R+, R-)$ we instead have massless **bosonic** fields with two spinor indices, one chiral and one anti-chiral: Denote these as $A_{a\dot{b}}$.

Recall that for the gamma matrices related to $SO(1,3)$ (the ordinary Lorentz group) there is a basis of 16 matrices $\gamma^{[n]}$, $n = 0,1,2,3,4$. Similarly for $SO(8)$ the basis of 16×16 matrices is $\Gamma^{[n]}$, $n = 0,1,...,8$. However, we need matrices with chiral indices so we must consider the four 8×8 blocks separately. This is very similar to how we can express $SO(1,3)$ Dirac matrices in block form using Pauli matrices. Without providing a proof we just claim that the chiral expansion is

$$\text{Type IIA:}\;\; A_{a\dot{b}} = (\Gamma^I)_{a\dot{b}}A_I + \frac{1}{3!}(\Gamma^{IJK})_{a\dot{b}}A_{IJK}. \tag{10.56}$$

We conclude that this spectrum contains the massless bosonic tensor fields A_μ and $A_{\mu\nu\rho}$. The count of degrees of freedom follows directly from the light-cone since both fields are gauge fields : A_μ has $D-2=8$ d.o.f. and $A_{\mu\nu\rho}$ has $\frac{1}{3!}(D-2)D-3)D-4)=56$. So these two tensor fields have together 64 d.o.f. which together with the (NS_+, NS_+) gives in total 128 bosonic d.o.f. which is exactly the same as in the fermionic spectrum.

Type IIB: The analysis is similar to the one for type IIA above. Supersymmetry is now referred to as being $(2,0)$. The main difference is due to the chiral nature of the spinor indices: Having now two chiral indices of the same type the matrix expansion reads instead

$$\text{Type IIB:}\;\; A_{ab} = \delta_{ab}A + \frac{1}{2!}(\Gamma^{IJ})_{ab}A_{IJ} + \frac{1}{4!}(\Gamma^{IJKL})_{ab}A_{IJKL}. \tag{10.57}$$

The d.o.f. count in $D=10$ is then 1, $\frac{1}{2}(D-2)(D-3)=28$, and $\frac{1}{4!}(D-2)(D-3)(D-4)(D-5)=70$. Since the sum must be the same as for type IIA there seems to be a problem here. However, the field A_{IJKL} is not an irreducible tensor since it can be split into a self-dual and an anti-self-dual piece using the 8-dimensional epsilon tensor

$$A^{\pm}_{IJKL} := \frac{1}{2}(\delta^{MNPQ}_{IJKL} \pm \frac{1}{4!}\epsilon_{IJKL}{}^{MNPQ})A_{MNPQ}. \tag{10.58}$$

Clearly we want only one of these two, having 35 d.o.f. each, to appear in the expansion above for A_{ab}. Fortunately, this is automatic since the two chiral spinor indices on $(\Gamma^{IJKL})_{ab}$ implies that this gamma matrix satisfies the above projection to the self-dual part. That is, only A^{+}_{IJKL} appears in the gamma-matrix expansion for A_{ab} above.

Let us summarise the massless part of the spectra of these two superstring theories with **two supersymmetries**

$$\text{Type IIA: } g_{IJ}, B_{IJ}, \phi, A_I, A_{IJK}, \psi_{aI}, \psi_{\dot{a}I}, \lambda_a,\ \lambda_{\dot{a}}, \tag{10.59}$$

$$\text{Type IIB: } g_{IJ}, B_{IJ}, \phi, A, A_{IJ}, A^{+}_{IJKL}, \psi^i_{aI}, \lambda^i_{\dot{a}}, \quad i = 1, 2. \tag{10.60}$$

The Lorentz covariant version of these fields is (the spinors are denoted by $(a, \dot{a})$ also here although they have twice as many components as in the light-cone)

$$\text{Type IIA: } g_{\mu\nu}, B_{\mu\nu}, \phi, A_\mu, A_{\mu\nu\rho}, \psi_{a\mu}, \psi_{\dot{a}\mu}, \lambda_a,\ \lambda_{\dot{a}}, \tag{10.61}$$

$$\text{Type IIB: } g_{\mu\nu}, B_{\mu\nu}, \phi, A, A_{\mu\nu}, A^{+}_{\mu\nu\rho\sigma}, \psi^i_{a\mu}, \lambda^i_{\dot{a}}, \quad i = 1, 2. \tag{10.62}$$

The Lagrangians for all these fields can in principle be derived from the corresponding string theory. However, these so called *low energy field theory* Lagrangians are extremely complicated and involve, for instance, terms with arbitrary high powers of gauge invariant field strengths, like the Riemann tensor, and only a few of the low-derivative terms are known.

In particular, the terms with no more than two derivatives are rather simple to write down which leads to the well-studied supergravity theories. This two-derivative part of the full low-energy field theory Lagrangian has relatively few terms which in the bosonic sector can only (with one exception) involve the field strengths

$$R_{\mu\nu\rho\sigma}, H_{\mu\nu\rho}, C_{\mu_1\ldots\mu_{n+1}}. \tag{10.63}$$

These are constructed as usual from the metric, the Kalb-Ramond field $B_{\mu\nu}$ and the anti-symmetric tensor fields $A_{\mu_1\ldots\mu_n}$. There are in the Lagrangian some additional interesting features, e.g., 1) the dilaton enters via e^{ϕ}, raised to various powers, multiplying the terms in the Lagrangian. 2) There are also terms of a more topological nature where a gauge field appears without derivatives (e.g., the term $C_4C_4B_2$ in type IIA).

Comment: Although the Lagrangians of all the supergravity theories discussed above contain a number of different terms they are of a rather conventional type. The 11-dimensional M-theory field theory Lagrangian is presented below as an example. Two particularly important and less conventional features appear, however, in the type *IIB* Lagrangian:

1) This theory contains two scalar fields, the scalar $\phi(x)$ and the pseudo-scalar $A(x)$. They appear in the Lagrangian together as a **non-linear sigma-model**, i.e., these two fields can be viewed as coordinates on the coset space $SU(1,1)/U(1)$ (which is an unbounded Euclidean hyperbolic space familiar from the course in GR).

2) A unique feature of IIB is that it contains the anti-symmetric four-indexed field $A^+_{\mu\nu\rho\sigma}$. This field is nicely defined in the light-cone as self-dual in the four indices (indicated by the $+$). However, in the covariant formulation the Lagrangian must be written in terms of its field strength $G^+_{\mu_1...\mu_5}$ also anti-symmetric in all its indices. Then one can see from the field equations that the self-duality must be imposed on this field strength. Strangely enough, this field equation can not be derived from a Lagrangian since $\int d^{10}x\sqrt{-g}G^+_{\mu_1...\mu_5}G^{+\mu_1...\mu_5} = \int d^{10}x\sqrt{-g}\epsilon^{\mu_1....\mu_{10}}G^+_{\mu_1...\mu_5}G^+_{\mu_6...\mu_{10}} = 0$. This problem is still not completely resolved!

The tensor fields are rather simple generalisations of the Maxwell theory and it will be very important to understand the meaning of charge, both electric and magnetic, in relation to these new anti-symmetric tensor field theories. Developing a good understanding of these charges will give us a path into the theory of **branes** in string theory.

Other string theories:
In many popular accounts of string/M-theory one organises all the different superstring and M-theories, or rather their low energy supergravities, by associating them with the cusps of what we may call the **amoeba diagram**: see E. Witten, in Physics Today, May 1997. (A more updated popular account of string theory and quantum gravity can be found in E. Witten, Physics Today, November 2015.)

Of the six low energy theories at the cusps we have studied only two so far: **type IIA** and **type IIB**. These two are related by a so called **T-duality** which will be studied in detail later in this course. The type IIA supergravity theory in 10 spacetime dimensions can be more efficiently described in 11 dimensions by relating the string dilaton (and hence the string coupling constant) to the size of a compact (S^1) extra tenth space direction. Turning this argument around, we can start by constructing 11-dimensional supergravity (the perhaps simplest of all such theories) and then compactify its tenth space direction on a circle with radius R_{11} given by $R_{11} = g_{10,10}$. In ten dimensions, the field R_{11} defined this way in 11 dimensions turns out to be related to the dilaton in string theory: The compactification gives the relation, first found by Ed Witten in the 1990s,

$$2\pi R_{11} = g_s^{2/3}\, l_p, \tag{10.64}$$

where l_p is the Planck length and $g_s := e^{\langle\phi\rangle}$ the usual string coupling constant.

There is a very peculiar implication of the relation $2\pi R_{11} = g_s^{2/3}\, l_p$. From the point of view of type IIA in ten dimensions, turning the coupling constant g_s up so that the theory becomes very strongly coupled also means that the new direction looks more and more like an infinite direction on equal footing with the uncompactified directions: The theory turns into an 11-dimensional theory. This is what **duality** is all about: For small g_s the type IIA theory can be studied using perturbation theory in ten spacetime dimensions while for big g_s the strongly coupled IIA theory has no well-defined perturbation theory in 10 dimensions but after the duality transformation to 11 dimensions it can again be studied perturbatively (now in $G_N^{(11)}$).

Comment: 11-dimensional supergravity (see below) is related to M-theory (as a theory of M-branes) as $D = 10$ supergravity is related to string theory, that is as a low energy approximation. However, since M-theory has no scalar field in the massless spectrum there does not seem to exist a coupling constant similar to g_s in string theory. This fact means, according to many people in the field, that the full M-theory derived from M2 and M5 branes (discussed later in the course if time permits) is non-perturbative (i.e., there does not exist any perturbative definition of it similar to the string loop-expansion). This makes this theory extremely interesting but also extremely hard to study.

11-dimensional supergravity has only three fields

$$\text{M-theory: } g_{\mu\nu},\ A_{\mu\nu\rho},\ \psi_\mu. \tag{10.65}$$

The action is

$$S[g_{\mu\nu}, A_{\mu\nu\rho}, \psi_\mu] = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-g}(R - \frac{1}{2\cdot 4!}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} - \frac{i}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho) \tag{10.66}$$

$$- \frac{1}{(12)^4}\frac{1}{2\kappa_{11}^2}\int d^{11}x\, \epsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\mu_{10}\mu_{11}}) + \ldots. \tag{10.67}$$

1) The first line contains the standard kinetic terms for the three M-theory fields: The Einstein-Hilbert term for gravity, the Kalb-Ramond field strength squared and spin 3/2 generalisation of the usual Dirac term for spin 1/2 fermions.
2) The second line is a kind of topological term since it involves an epsilon tensor density and is actually independent of the metric (and hence of the spacetime geometry[29]).

There are several other terms but they are all interaction terms involving $\bar{\psi}_{\mu_1}\Gamma^{\mu_1\ldots\mu_6}\psi_{\mu_2}$ and other similar spinor combinations, one example being

$$\propto \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-g}(\bar{\psi}_{\mu_1}\Gamma^{\mu_1\ldots\mu_6}\psi_{\mu_2} + 12\bar{\psi}^{\mu_3}\gamma^{\mu_4\mu_5}\psi^{\mu_6})(F_{\mu_3\ldots\mu_6} + \tilde{F}_{\mu_3\ldots\mu_6}). \tag{10.68}$$

This expression is an interaction term between three fields (and of course the metric) and is a rather simple expression. There are, however, some implicit definitions that complicates the theory a bit. These aspects will not be explained here[30].

[29] In differential geometry we would express it is an 11-form $\int F_4 \wedge F_4 \wedge A_3$.
[30] For more details, see Duff, Nilsson and Pope, "Kaluza-Klein supergravity", Physics Reports, Vol. 130 (1986), p. 1 - 142 or Becker, Becker and Schwarz, "String theory and M-theory" (CUP 2007).

The local supersymmetry is given by (here α is a flat vector index in $D = 11$)

$$\begin{aligned} \delta e_\mu{}^\alpha &= \bar\epsilon \Gamma^\alpha \psi_\mu, \quad \delta A_{\mu\nu\rho} = -3\bar\epsilon \Gamma_{[\mu\nu}\psi_{\rho]}, \\ \delta\psi_\mu &= \nabla_\mu \epsilon + \frac{1}{24}(\Gamma_\mu{}^{\nu\rho\sigma\tau} - 8\delta_\mu^{[\nu}\Gamma^{\rho\sigma\tau]})\epsilon F_{\nu\rho\sigma\tau}. \end{aligned} \tag{10.69}$$

Kaluza-Klein compactification of 11D supergravity to type IIA in 10D
It is rather easy to understand the close Kaluza-Klein connection between 11-dimensional supergravity and type IIA supergravity in 10 dimensions. Let us denote indices in 11D by capital $M, N, ..$ and in 10D as usual by $\mu, \nu, ...$ Then compactification on a circle implies

$$g_{MN} \to (g_{\mu\nu}, A_\mu, \phi), \;\; A_{MNP} \to (A_{\mu\nu\rho}, B_{\mu\nu}). \tag{10.70}$$

Spinorial indices behave in a different way under Kaluza-Klein compactification. Let $\bar a$ be a 32-component spinor index in 11D. Then the spin 3/2 Rarita-Schwinger field in 11D splits into 10-dimensional fields as follows

$$\psi_{\bar a, M} \to (\psi_{a\mu}, \;\psi_{\dot a\mu}, \;\lambda_a, \;\lambda_{\dot a}), \tag{10.71}$$

where a and $\dot a$ are the two chiral, left and right, spinors in 10D. Thus we see that 11D supergravity reducers exactly to the *non-chiral* type IIA theory in 10D.

Type IIA to type IIB: This is achieved by T-duality which is a phenomenon we will study later.

From type IIB to type I supergravity
There is also another kind of string theory in 10 dimensions which has only one Rarita-Schwinger field and hence is $\mathcal{N} = 1$ supersymmetric. This theory is obtained from the type IIB theory by imposing invariance under **orientation reversal** on the closed string. This restricts the spectrum to symmetric tensors in the $(NS+, NS+)$ sector and to antisymmetric ones (in spinor indices) in the $(R-, R-)$ sector. The invariant part of the IIB spectrum defines the **type I**, or $(1, 0)$, superstring theory which then has the spectrum $g_{\mu\nu}, \phi, A_{\mu\nu}, \psi_\mu$.

This theory turns out to be ill-defined (it has anomalies) which is fixed by adding a set of open strings to the closed string theory. In the low energy field theory this open string sector generates a super-Yang-Mills theory with gauge group $SO(32)$. Thus we have ($i = 1, ..., 496$, the dimension of $SO(32)$)

$$\text{Type I:} \;\; g_{\mu\nu}, \phi, A_{\mu\nu}, \psi_\mu, A_\mu^i, \lambda^i. \tag{10.72}$$

Heterotic strings:
Returning to M-theory Witten discovered, together with Horava, in 1996 that it is possible to compactify the 11th direction on a segment. This means that the 10-dimensional

spacetime has two branches, or planes, which must each contain not only the compactified fields from 11D but also (to cancel anomalies) a super-Yang-Mills theory with gauge group E_8. This is called the $E_8 \times E_8$ **heterotic** string theory[31]. It is an $\mathcal{N} = 1$ closed string theory which has a very intricate way of giving rise to the Yang-Mills gauge theory (see the discussion of vertex operators below). The spectrum of the heterotic $E_8 \times E_8$ low energy field theory is:

$$\text{The } E_8 \times E_8 \text{ heterotic string: } g_{\mu\nu}, B_{\mu\nu}, \phi, \psi_\mu, A^i_\mu, \lambda^i \text{ with gauge group } E_8 \times E_8. \quad (10.73)$$

Performing a T-duality (defined below) on the this theory produces a different theory of a similar kind, namely one with an $SO(32)$ gauge group,

$$\text{The } SO(32)/\mathbf{Z_2} \text{ heterotic string: } g_{\mu\nu}, B_{\mu\nu}, \phi, \psi_\mu, A^i_\mu, \lambda^i \text{ with gauge group } SO(32)/\mathbf{Z_2}. \quad (10.74)$$

Comment: The string (Kac-Moody) construction of the two heterotic theories is heavily based on the theory of Euclidean even self-dual lattices which exist only in $8n$ dimensions[32] The single such lattice that exists in 8 dimensions is related to the exceptional Lie group E_8 (it is the root lattice) and in 16 dimensions the two existing lattices are related to the gauge groups of the two heterotic string theories above. The connection to these self-dual lattices arises at the string 1-loop level.

Dualities: This completes the description of the low energy supergravity theories at the six nodes (or cusps) of the amoeba diagram. The edges between the cusps represent the different duality transformations that can be used to transform the adjacent theories into each other, basically **T-duality** (inversion of a radius) and **S-duality** (inversion of a coupling constant)[33]. It is therefore possible to start from the 11-dimensional supergravity given above and derive all the string theory related supergravities around the amoeba diagram using dualities of various kinds. This clearly gives the **M-supergravity** theory a very special status among all these theories. Note that we did not get all around the amoeba diagram above: The connection between the type I and the heterotic $SO(32)$ theories was not provided. However, this is done using S-duality. A last quite interesting duality is the S-duality in the type IIB case: It takes the theory to itself!

M-theory

The key question is however: What is the theory in the middle of the amoeba diagram? In that area we have moved away from the cusps and the well-behaved perturbative supergravity/string theories into a region of parameter space where the master M-theory (now based on M-branes instead of strings) is probably strongly coupled. In this region very little is known about the theory. M-branes are of two kinds, $M2$ and $M5$, which appear as generalised multi-dimensional black hole solutions of the field equations coming from the

[31] This closed string theory is built from a compactified bosonic string for the left movers and a superstring for the right movers.

[32] "Even" means that all vectors in the lattice has length-square equal to an even integer.

[33] By combining T and S duality a more complicated duality structure may appear called U-duality.

11-dimensional field theory Lagrangian given above. The D-branes to be analysed later are string analogues of these M-branes.

The M2 and M5 branes:

We end this discussion by explaining the nature of these M-branes. Recall first the **Schwarzschild black hole** solution (BH) of the vacuum Einstein equations $R_{\mu\nu} = 0$ in 3+1 dimensions. This solution was derived very carefully in the gravity course:

$$ds^2(BH) = -(1-\frac{r_0}{r})dt^2 + (1-\frac{r_0}{r})^{-1}dr^2 + r^2 d\Omega_2^2, \quad r_0 = 2GM. \tag{10.75}$$

We see here that the solution approaches Minkowski as $r \to \infty$ and that there is a horizon at $r = r_0 = 2GM$. It is sometimes useful to change to a radial coordinate ρ for which $\rho = 0$ is the horizon: $r = r_0 + \rho$. Then $1 - \frac{r_0}{r} = \frac{\rho}{r_0+\rho} = 1/(1+\frac{r_0}{\rho})$ so the Schwarzschild metric becomes

$$ds^2(BH) = -(1+\frac{r_0}{\rho})^{-1}dt^2 + (1+\frac{r_0}{\rho})dr^2 + (r_0+\rho)^2 d\Omega_2^2, \quad r_0 = 2GM. \tag{10.76}$$

There is a completely analogous solution if the we instead consider a *charged* black hole (QBH) by solving the coupled Einstein-Maxwell equations in 3+1 dimensions:

$$ds^2(QBH) = -(1-\frac{r_0}{r}+\frac{Q^2G}{r^2})dt^2 + (1-\frac{r_0}{r}+\frac{Q^2G}{r^2})^{-1}dr^2 + r^2 d\Omega_2^2, \quad r_0 = 2GM. \tag{10.77}$$

and, with the only non-zero $F_{\mu\nu}$ component,

$$F_{tr} = E_r = \frac{Q}{r^2}. \tag{10.78}$$

This so called **Reissner-Nordström** solution will be the prototype for all the supergravity solutions that we will later claim (without proof in this course) correspond to the various D-branes discussed in Part 2 of the course. An interesting fact about this solution is that it has **two** horizons given by

$$r = r_\pm = MG \pm \sqrt{(MG)^2 - Q^2G}. \tag{10.79}$$

Obviously $(MG)^2 \geq Q^2G$ which implies that this black hole can loose energy by Hawking radiation until $(MG)^2 = Q^2G$ after which it becomes stable[34]. This stable solution is called the **extremal** solution[35].

Turning now to supergravity in $D = 11$ spacetime dimensions we can try to find similar solutions of the coupled Einstein-Kalb-Ramond field equations. The two solutions that exist have two and five space dimensions, respectively, for the $M2$ and the $M5$ branes. The *extremal* versions of these solutions are

$$M2: \;\; ds^2 = H^{-\frac{2}{3}}dx^2 + H^{\frac{1}{3}}dy^2, \quad H(r) = 1 + \frac{r_2^6}{r^6}, \;\; r_2^6 = 32\pi^2 N_2 l_p^6, \tag{10.80}$$

[34]This is connected to the fact that this background is compatible with supersymmetry.
[35]It is also known as a *BPS* solution.

$$F_{012r} \propto \partial_r \frac{1}{H(r)}, \tag{10.81}$$

and

$$M5: \;\; ds^2 = H^{-\frac{1}{3}} dx^2 + H^{\frac{2}{3}} dy^2, \;\;\; H(r) = 1 + \frac{r_5^3}{r^3}, \;\; r_5^3 = \pi^2 N_5 l_p^3, \tag{10.82}$$

$$F_{012345r} \propto \partial_r \frac{1}{H(r)}, \;\;\; F_{M_1..M_4} = \frac{1}{7!} \epsilon_{M_1..M_4}{}^{M_5.....M_{11}} F_{M_5.....M_{11}}. \tag{10.83}$$

Here x^μ are the coordinates on the flat Lorentzian branes and y^m are the remaining coordinates in 11D. The radius r is the distance away from the brane in these off-brane dimensions. The number N_2 and N_5 are the brane charges corresponding to stacks with these number of branes (the branes are normally given a unit charge). This interpretation comes out when comparing these solutions to the construction of branes from open strings. We will have reason to return to this relation between solutions and branes in the context of AdS/CFT later.

The Polyakov supergravity theory

As promised above, we will now return to the Lorentz covariant formulation of the superstring. The relevant theory is then the locally supersymmetric version of the Polyakov action discussed before in the bosonic case. The local supersymmetry is quite similar to the global one discussed in detail above, but a bit more complicated since new terms arise in the super-Polyakov Lagrangian for basically two different reasons:
1) From the supersymmetry parameters which are now functions on the world-sheet and
2) from the two-dimensional supergravity fields, metric $h_{\alpha\beta}$ and the Rarita-Schwinger field χ_α (which has a world-sheet spinor index that is not written out).

The action is[36]

$$S = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-h}\left(\frac{1}{2}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} + \frac{i}{2}\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu} - \frac{i}{2}\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu (\partial_\beta X^\nu - \frac{i}{4}\bar{\chi}_\beta \psi^\nu)\eta_{\mu\nu}\right). \tag{10.84}$$

This action is invariant under local supersymmetry transformations with spinor parameter $\epsilon(\tau,\sigma)$ given on the string super-coordinates (X^μ, ψ^μ) by

$$\delta_\epsilon X^\mu = i\bar{\epsilon}\psi^\mu, \tag{10.85}$$

$$\delta_\epsilon \psi^\mu = \rho^\alpha \epsilon\,(\partial_\beta X^\nu - \frac{i}{2}\bar{\chi}_\beta \psi^\nu). \tag{10.86}$$

Requiring this theory to be invariant under both local coordinate and local supersymmetry transformations implies that the world-sheet supergravity fields must transform as follows:

$$\delta_\epsilon\, e_\alpha{}^a = i\bar{\epsilon}\rho^\alpha \chi_\alpha, \tag{10.87}$$

$$\delta_\epsilon \chi_\alpha = 2D_\alpha \epsilon := 2(\partial_\alpha \epsilon - \frac{1}{2}\omega_\alpha \rho^3 \epsilon). \tag{10.88}$$

Without going into the details we just note that since we are dealing with spinors on a curved world-sheet one has to introduce a zweibein field $e_\alpha{}^a(\tau,\sigma)$ which relates the coordinate basis and an arbitrary orthogonal basis on the world-sheet tangent space (so the index a is here a 2-dimensional Lorentzian vector index). Its relation to the metric is given by $h_{\alpha\beta} = e_\alpha{}^a e_\beta{}^b \eta_{ab}$. Having introduced this orthogonal basis the Lorentz symmetry must be gauged which is done by introducing the spin-connection field denoted ω_α in the $\delta_\epsilon \chi_\alpha$ transformation above. This means that D_α is a Lorentz covariant derivative.

This covariant action is of course globally Poincaré invariant but it has a number of important local symmetries generalising the ones in the bosonic case. These local symmetries are world-sheet reparametrisations, local supersymmetry, local Lorentz transformations plus the local scale and local super-scale invariances

$$\text{Weyl:}\ \delta\, e_\alpha{}^a = \Lambda e_\alpha{}^a,\ \ \delta\chi_\alpha = \tfrac{1}{2}\Lambda\chi_\alpha,\ \ \delta\psi^\mu = -\tfrac{1}{2}\Lambda\psi^\mu,\ \ (X^\mu \text{ are inert}), \tag{10.89}$$

[36]For details and the original work, see Brink, Di Vecchia and Howe, Physics Letters 65B (1976) p. 471-474.

$$\text{super-Weyl: } \delta\chi_\alpha = \rho_\alpha \eta, \text{ (all other fields are inert).} \tag{10.90}$$

As in the bosonic case we can use these local symmetries to choose the **superconformal gauge** defined by

$$h_{\alpha\beta} = \rho^2(\tau,\sigma)\eta_{\alpha\beta}, \tag{10.91}$$

where $\rho^2(\tau,\sigma)$ is a local scale factor (i.e., not gamma matrices), and

$$\chi_\alpha = \rho_\alpha \lambda(\tau,\sigma), \tag{10.92}$$

where ρ_α are gamma matrices and $\lambda(\tau,\sigma)$ a local spinor (similar to the scale factor above).

Using these gauge conditions the Polyakov supergravity action above simplifies directly to the super-Polyakov action on a flat world-sheet which we proved earlier to be invariant under global supersymmetry transformations.

However, before implementing the gauge conditions we should compute the field equations for the supergravity fields. So let us vary the action with respect to $\delta h_{\alpha\beta}$ and $\delta\chi_\alpha$ which will give us the Einstein equations (without the Einstein tensor term as in the bosonic case) and the Rarita-Schwinger equation (without the Rarita-Schwinger kinetic term). Then we implement the superconformal gauge on these equations. This gives the following nice results

$$\delta h_{\alpha\beta} \Rightarrow \quad T_{\alpha\beta} = 0 \text{ where}$$

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} + \frac{i}{2}\bar{\psi}^\mu \rho_{(\alpha}\partial_{\beta)}\psi^\nu \eta_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}(\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} + \frac{i}{2}\bar{\psi}^\mu \rho_{(\gamma}\partial_{\delta)}\psi^\nu \eta_{\mu\nu}), \tag{10.93}$$

which we now see contains two new terms from the fermionic part of the Lagrangian.

We also find an entirely new spinorial current, the supercurrent J^α,

$$\delta\chi_\alpha \Rightarrow \quad J_\alpha = 0 \text{ where } J^\alpha = \frac{1}{2}\rho^\beta \rho^\alpha \psi^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{10.94}$$

These two objects are both Noether currents and are thus on-shell conserved

$$\partial_\alpha T^{\alpha\beta} = 0, \;\; \partial_\alpha J^\alpha = 0, \tag{10.95}$$

which follow from reparametrisation invariance and supersymmetry. However, the Weyl and super-Weyl symmetries imply that these currents also satisfy

$$T^\alpha{}_\alpha = 0, \;\; \rho_\alpha J^\alpha = 0. \tag{10.96}$$

This can be verified from the expressions above for these currents. When computing the currents from the action above these conditions should be checked to avoid mistakes.

The above forms of the stress tensor $T_{\alpha\beta}$ and the supercurrent J_α explain what is happening in the transition from the bosonic string to the superstring. Using the light-cone coordinates (σ^+, σ^-) on the world-sheet the superstring stress tensor components read

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_+, \tag{10.97}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_-. \tag{10.98}$$

Without going into details we can see that an expansion of the stress tensor into Virasoro generators L_n implies that they now contain contributions from both the bosonic modes, α_n^μ, and the fermionic ones, d_n^μ or b_r^μ. This is in accord with our expressions for the mass operator M^2 presented previously.

The new anti-commuting current, the **supercurrent** J_α, has light-cone components

$$J_+ = \psi_+^\mu \partial_+ X_\mu, \tag{10.99}$$

$$J_- = \psi_-^\mu \partial_- X_\mu. \tag{10.100}$$

The constraints that must be imposed on the state space for the superstring are

$$T_{++} = T_{--} = J_+ = J_- = 0. \tag{10.101}$$

Also the supercurrents must be expanded in modes. These modes are called G_r in the NS sector and F_n in the R sector. Together with the superstring Virasoro generators, all these generators give rise to the super-Virasoro algebra containing both commutators and anti-commutators. As an example how the new supergenerators can be used, consider

$$F_0 = \alpha_0^\mu d_{0\mu} + \alpha_{-1}^\mu d_{1\mu} + \alpha_1^\mu d_{-1\mu} + ... \tag{10.102}$$

Just as we impose the primary state condition $(L_0 - 1)|phys\rangle = 0$ in the bosonic string we must in the superstring impose also $F_0|phys\rangle = 0$ in the R sector. If we consider the R ground states $|R_a\rangle_{(R)}$ only the first term in F_0 above is non-zero (since for $n > 0$ we have $\alpha_n^\mu|R_a\rangle_{(R)} = d_n^\mu|R_a\rangle_{(R)} = 0$) and the condition becomes if written in matrix form

$$\Gamma^\mu p_\mu |R_a; p\rangle = 0. \tag{10.103}$$

This is the Fourier transform of the spacetime Dirac equation $\Gamma^\mu \partial_\mu \psi(x) = 0$.

With this overview of superstring theory we end Part 1, the introductory *Basic* part, of the course and turn to Part 2, *Developments*.

11 Lecture 13

In this lecture we will study the antisymmetric tensor fields that are part of the spectrum of the various string theories discussed in previous lectures. Since some of these fields have more than one antisymmetric vector index they can be viewed as generalisations of the Maxwell vector potential, and their electric and magnetic charges can be discussed in similar terms. Thus it is important to study the objects that carry these charges, i.e., the D-branes. As we will see later, stacks of such branes are instrumental in deriving Standard Model like physics from strings.

11.1 Chapter 15: D-branes

In the following discussions we will always consider strings and branes that come from a superstring and hence live in $D = 10$ spacetime dimensions. However, most of the considerations involve only the bosonic sector of these theories.

Recall first how we used a table to describe open string boundary conditions in a previous lecture. Here we will use a similar table that is useful for D-branes living in 9 space dimensions. As an example consider the following table for the space components X^i:

i= End	1	2	3	4	5	6	7	8	9
$\sigma = 0$	D	D	D	N	N	N	N	N	N
$\sigma = \pi$	D	D	D	N	N	N	N	N	N

This table represents open strings with both ends on a D-brane spanned by the space directions 4 to 9 (the N bc directions) and they are hence unable to move perpendicularly to these directions (i.e., in the directions 1,2 and 3). This is a Dp-brane with $p = 6$.

So to understand the spectra arising from strings involved in this example we need mode expansions for both (N, N) and (D, D) b.c.s. The former case was obtained in a previous lecture (now with the spacetime direction suppressed)

$$(N, N): \quad X(\tau, \sigma) = x_0 + 2\alpha' p\tau + i\sqrt{2\alpha'}\, \Sigma_{n\neq 0}^{\infty} \frac{1}{n}\alpha_n e^{-in\tau} \cos n\sigma. \tag{11.1}$$

The (D, D) situation has not been analysed before so this needs to be done now. However, it is rather trivial to see that the answer is, if the fixed direction is located at $x_0 = \bar{x}$,

$$(D, D): \quad X(\tau, \sigma) = \bar{x} + \sqrt{2\alpha'}\, \Sigma_{n\neq 0}^{\infty} \frac{1}{n}\alpha_n e^{-in\tau} \sin n\sigma. \tag{11.2}$$

There are some properties that require an explanation:
1. The operator zero modes (x_0, p) in the (N, N) are replaced in the (D, D) case by $\bar{x}$ which is a fixed number and not an operator. This fact is related to the non-appearance of a momentum term in the expansion. This is natural since the string ends cannot move in this direction (i.e., the direction of X).

2. The oscillator terms must vanish at both ends which is guaranteed by the $\sin n\sigma$ function. The $\sin n\sigma$ also implies reality of X without an i in front of the oscillator terms.

If there are two parallell D-branes involved in the setup, one at $\bar{x}_1$ and the other at $\bar{x}_2$, then the expansion for strings with one end on each brane, becomes

$$(D,D): \ \ X(\tau,\sigma) = \bar{x}_1 + (\bar{x}_2 - \bar{x}_1)\frac{\sigma}{\pi} + \sqrt{2\alpha'}\,\Sigma_{n\neq 0}^{\infty}\frac{1}{n}\alpha_n e^{-in\tau}\sin n\sigma. \tag{11.3}$$

Note that the second term is not a momentum term since it contains σ, not τ.

It is important to realise that the mass spectrum is effected by these Dirichlet boundary conditions:

$$M^2 = \left(\frac{\bar{x}_2 - \bar{x}_1}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N^{\perp} - 1). \tag{11.4}$$

Let us now discuss some of the physical implications of this formula, including one particularly important one which is hidden in the notation used. First, the first term depends on the distance between the D-branes. This term comes from the σ dependence of the corresponding term in the mode expansion. It will survive the construction of the Virasoro generators (from X' in $\dot{X} \pm X'$) and since it is not a momentum term it is not part of $M^2 = 2p^+p^- - p^Ip^I$ and therefore ends up on the RHS.

Secondly, what is part of $M^2 = -p^{\mu}p_{\mu} = 2p^+p^- - p^Ip^I$ are all the momenta which, however, are only non-zero in the D-brane directions. Therefore, the most important aspect of the open string formula above is that the mass spectrum concerns only fields that "live on the D-brane", i.e., they don't depend on any coordinates spanning directions that are not along the D-brane.

In order to be very clear about what is being said above, we summarise it as follows:

D-brane physics: *The low-energy field theory generated by open strings with their end-points on some (parallell) Dp-branes live on these D-branes. These field theories are typically supersymmetric Yang-Mills theories (see below) and do NOT contain gravity. Obviously, since closed strings have nothing to do with D-branes their low-energy field theories (including gravity) live in all $D = 10$ dimensions of spacetime. Note that the open string oscillations can take place in all nine space directions even if the two end-points are tied to the one or two Dp-branes.*

Question: If the two D-branes are separated along some axis, which of the two D-branes does the field theory live on? (Tricky issue!)

Stacks of D-branes: Piling up a stack of D-branes (all parallell then) gives rise to new properties of the low-energy field theory, namely interactions. To see this consider three D-branes, called 1, 2 and 3. The open strings that can be associated with these branes must either have their two ends on the same D-brane, denoted $[ii]$-strings, or have the end-points tied to different D-branes, then denoted $[ij]$-strings (with $i \neq j$). Now consider two strings of this latter type, [12] and [23] say. The former clearly has its $\sigma = \pi$ end on brane 2 and the latter string has its $\sigma = 0$ end also on brane 2. If these two ends on brane 2 meet they can form a new longer open string which we denote as $[12] \star [23]$. Since the end-points on brane 2 have now been cancelled nothing forces any point on this long string to stick to brane 2 and it can hence just disconnect from it. Therefore we have shown that one can realise an open "2-strings $\rightarrow$ 1-string interaction" as follows:

$$[12] \star [23] \rightarrow [13]. \tag{11.5}$$

This transition will be represented in the field theory by two vector fields interacting to form just one vector field. This is precisely what the 3-point interaction in Yang-Mills theory is doing. In this field theory this interaction has a momentum dependence but it is easily checked that also string theory produces this momentum dependence (we need unfortunately some methods not developed in this course to prove this fact).

Let us now try to pinpoint exactly what kind of Yang-Mills theory is produced by open strings and D-branes. We start by considering an N-stack, that is N parallell D-branes on top of each other. This stack will produce N different open strings having both ends on the same brane and $N(N-1)$ strings with their ends on different branes. Note that there is no factor of 1/2 since the open strings are oriented. Together this makes N^2 different open string configurations, each one generating a massless vector gauge field. This suggests that we are dealing with an $U(N)$ Yang-Mills theory (which can be proven in full detail with other methods). However, from group theory we know that $U(N) = SU(N) \times U(1)$ which is the final answer.

One nice aspect of this result about the gauge theory on a stack of D-branes is that if one (or several) of them is moved away from the other branes some of the massless vector fields become massive and the gauge groups breaks down to a smaller group. It can in fact be shown that there are scalar fields involved in this process that are "eaten" and the whole process actually corresponds to the usual Higgs effect. For instance, for $N = 2$ the gauge group is $U(2) = SU(2) \times U(1)$ and the Higgsing obtained by separating the two D-branes generates two massless abelian gauge fields and two massive vector fields (corresponding to $W^{\pm}$ in the Standard Model).

Orthogonal/intersectiong D-branes: A final example that will be analysed here concerns D-branes at an angle, in fact orthogonal ones. Other cases of this kind will be very important later. An example is given by the following table, with both branes containing the origin $x^1 = ... = x^9 = 0$,

i= End	1	2	3	4	5	6	7	8	9
$\sigma = 0$	N	N	N	N	D	D	D	D	D
$\sigma = \pi$	D	D	D	N	N	N	N	N	N

This configuration of D-branes consists of one D4-brane in the direction 1 - 4 and one D6-brane in the directions 4 - 9. They have one direction, direction 4, in common so they are orthogonal in this multi-dimensional sense. This case contains a new kind of open strings, namely strings with one end having D bc and one end having N bc. The mode expansion for such an open string reads, if the $\sigma = 0$ end has N bc,

$$(N, D): \quad X(\tau, \sigma) = \bar{x}_2 + i\sqrt{2\alpha'}\,\Sigma_{r \in \mathbf{Z}+\frac{1}{2}}\, \alpha_r \,\frac{1}{r}\, e^{-ir\tau}\, \cos(r\sigma). \tag{11.6}$$

and for the other case

$$(D, N): \quad X(\tau, \sigma) = \bar{x}_1 + \sqrt{2\alpha'}\,\Sigma_{r \in \mathbf{Z}+\frac{1}{2}}\, \alpha_r \,\frac{1}{r}\, e^{-ir\tau}\, \sin(r\sigma). \tag{11.7}$$

These expansions are also needed for parallell D-branes where the branes have different dimensionality.

A note on notation: When mode expanding X^μ in cases where all three different kinds of b.c.s occur, e.g., (N, N), (D, D) and (N, D), one sometimes need to indicate this by giving the indices different names. We may e.g. use q, r, a (as in BZ)

$$X^\mu: \ (N, N) \to X^q, \ (N, D) \to X^r, \ (D, D) \to X^a. \tag{11.8}$$

11.2 Chapter 16: String charge and electric and magnetic D-brane charges

Recall the action for a charged particle coupled to a dynamical EM vector field

$$S[A_\mu, X^\mu] = -m\int_{\mathcal{P}} ds + q\int_{\mathcal{P}} A - \frac{1}{4\kappa_0^2}\int d^D x F_{\mu\nu}F^{\mu\nu}, \tag{11.9}$$

where $\mathcal{P}$ is the path, or world-line, of the particle in Minkowski space given by $X^\mu(\tau)$ (for some arbitrary parameter τ), $ds = \sqrt{-\eta_{\mu\nu}dX^\mu dX^\nu} = \sqrt{-\dot{X}^\mu\dot{X}^\nu}d\tau$ and we have introduced the parameter κ_0 with some dimension to make the action dimensionless (in natural units) for any spacetime dimension D while using fields A_μ with dimension $1/L$ as in $D = 4$. Finally, $F_{\mu\nu}$ is gauge invariant under $\delta A_\mu = \partial_\mu\Lambda$ as usual.

Consider now the coupling term (recall that a 1-form $A := dx^\mu A_\mu(x)$)

$$q\int_{\mathcal{P}} A := q\int_{\mathcal{P}} dX^\mu A_\mu(X) = q\int_{\mathcal{P}} \dot{X}^\mu(\tau)A_\mu(X(\tau))d\tau = q\int_{\mathcal{P}} d\tau\dot{X}^\mu(\tau)\int d^D x\,\delta^D(x-X(\tau))A_\mu(x). \tag{11.10}$$

From the definition $q\int_{\mathcal{P}} A := \int d^D x\, j^\mu(x)A_\mu(x)$ of the charged current for a point particle we get

$$j^\mu(x) = q\int_{\mathcal{P}} d\tau\dot{X}^\mu(\tau)\,\delta^D(x - X(\tau)). \tag{11.11}$$

Performing a $\delta A_\mu(x)$ variation we see that the coupling term $q\int_{\mathcal{P}} A$ contributes to Maxwell's equations and that it is gauge invariant under $\delta A_\mu = \partial_\mu\Lambda$ in the current is conserved $\partial_\mu j^\mu = 0$ (modulo boundary terms). This result follows also from the Maxwell equations.

The million dollar question is now: How do we generalise this to the string?

Consider the analogy between a charged point particle in an EM field and the string:

particle	q	A_μ	world-line	$q\int_{\mathcal{P}} A$	(11.12)
string	??	??	world-sheet	??	(11.13)

To fill in the question marks above we may recall the Polyakov action in a general background of the massless fields $g_{\mu\nu}(x)$ and $B_{\mu\nu}(x)$ (leaving $\phi(x)$ aside for the moment) in the closed string spectrum:

$$S[h, X] = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X) + 2\pi\alpha'\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X)\right), \tag{11.14}$$

where $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ is defined by $\epsilon^{01} = +1$. Concentrating on the second term we see that it is very similar in spirit to $q\int_{\mathcal{P}} A$: Define (note that here the definition of the 2-form contains the factor $\frac{1}{2}$ which differs from BZ)

$$S_B[X] := -\int_{\Sigma_2} B = -\frac{1}{2}\int_{\Sigma_2} dX^\mu dX^\nu B_{\mu\nu}(X) = -\int_{\Sigma_2} d\tau d\sigma\dot{X}^\mu X'^\nu B_{\mu\nu}(X). \tag{11.15}$$

The normalisation of the $B_{\mu\nu}$-field is now such that the 2-form B is dimensionless which means that the string has charge +1 in relation to the $B_{\mu\nu}$-field.

The field $B_{\mu\nu}$ has a gauge transformation $\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ which implies that the term in the Lagrangian corresponding to the Maxwell term $F^2_{\mu\nu}$ is the square of the field strength of $B_{\mu\nu}$:

$$H_3 = dB_2 \Rightarrow H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}. \tag{11.16}$$

There is of course also a Bianchi identity:

$$dH_3 = 0 \Rightarrow \partial_{[\mu}H_{\nu\rho\sigma]} = 0. \tag{11.17}$$

The coupled string-B-field action is then

$$S[X, B, H] = -\frac{1}{6\kappa_1^2}\int d^Dx\, H_{\mu\nu\rho}H^{\mu\nu\rho} + S[X, B], \tag{11.18}$$

where

$$S[X, B] = -\frac{1}{4\pi\alpha'}\int_{\Sigma_2} d\tau d\sigma \left(\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}(X) + 2\pi\alpha' \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X)\right). \tag{11.19}$$

We have then answered all the question marks above!

The string current have, however, some very unusual features. Similarly to the charged particle in an EM field we find now

$$\partial_\mu H^{\mu\nu\rho} = \kappa_2^2 j^{\nu\rho}, \tag{11.20}$$

$$j^{\mu\nu}(x) = \int_{\Sigma_2} d\tau d\sigma \delta^D(x - X(\tau, \sigma))\dot{X}^{[\mu}X'^{\nu]}. \tag{11.21}$$

Thus, obviously, this current is antisymmetric

$$j^{\mu\nu}(x) = -j^{\nu\mu}(x), \tag{11.22}$$

and satisfies a conservation equation (follows directly from the field equation and is similar to the one for the stress tensor)

$$\partial_\mu j^{\mu\nu} = 0. \tag{11.23}$$

It is at this point that the differences between EM and the string start to appear:

$$\text{EM}: \quad j^\mu = (j^0, j^i) = (\text{charge density, charge density current}), \tag{11.24}$$

$$\text{String}: \quad j^{\mu\nu} = (j^{0\nu}, j^{i\nu}) = (j^{0i}, j^{ij}) = (\text{vector charge density, vector charge current density}), \tag{11.25}$$

since for $j^{i\nu}$ the component $j^{i0} = -j^{0i}$. So the string charge is itself a vector (in space) and is defined to point from the $\sigma = 0$ end to the $\sigma = \pi$ end of the open string, or around the closed string in a specified direction. The unoriented string mentioned previously can hence not couple to the $B_{\mu\nu}$ field which therefore does not appear in the spectrum at all (a fact we know from before).

The conservation equations $\partial_\mu j^{\mu\nu} = 0$ for this string charge current reads in components

$$\partial_i j^{i0} = 0, \;\; \partial_t j^{0i} + \partial_j j^{ji} = 0. \tag{11.26}$$

The string vector charge itself is conserved

$$Q^i_{string} := \int d^d x \, j^{0i} \Rightarrow \dot{Q}^i_{string} = \int d^d x \, \partial_t j^{0i} = -\int d^d x \, \partial_j j^{ji} = 0. \tag{11.27}$$

We now express the conclusions in words:
The charge associated with a string is a conserved vector Q^i_{string}, similar to a current that never stops. This is true for a closed string as well as for an open string in which case the current must continue onto the D-branes.

Let us now study what happens when open strings end on D-branes. First we recall the gauge invariance in EM. The interaction term for a charged particle in an electromagnetic field is

$$S_{int} = q \int_{\mathcal{P}} A = q \int_{\mathcal{P}} X^\mu A_\mu \tau = \int d^D x j^\mu(x) A_\mu(x). \tag{11.28}$$

Since the other terms in the action are gauge invariant by themselves we must have $\delta_{gauge} S_{int} = 0$. Thus

$$\delta_{gauge} S_{int} = \int d^D x j^\mu(x) \delta A_\mu(x) = \int d^D x j^\mu(x) \partial_\mu \Lambda(x) = -\int d^D x \partial_\mu j^\mu(x) \Lambda(x) = 0, \tag{11.29}$$

and we find that the current is conserved as a consequence of the gauge invariance of the charge particle interaction with EM. This follows by neglecting the boundary terms.
Repeating this for the string is a very different story. Consider

$$S_B = -\int d\tau d\sigma \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} B_{\mu\nu}(X). \tag{11.30}$$

Performing a gauge transformation $\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ we get

$$\delta_{\Lambda_\mu} S_B = -\int d\tau d\sigma \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu). \tag{11.31}$$

Using $\partial_\tau = \dot{X}^\mu \partial_\mu$ this can written

$$\delta_{\Lambda_\mu} S_B = -\int d\tau d\sigma ((\partial_\tau \Lambda_\mu) \partial_\sigma X^\mu - (\partial_\tau X^\mu) \partial_\sigma \Lambda_\mu). \tag{11.32}$$

Integrating by parts to remove the derivatives from the parameters Λ_μ gives

$$\begin{aligned} \delta_{\Lambda_\mu} S_B = -\int d\tau d\sigma (&\partial_\tau (\Lambda_\mu \partial_\sigma X^\mu) - \Lambda_\mu \partial_\tau \partial_\sigma X^\mu \\ &- \partial_\sigma (\Lambda_\mu \partial_\tau X^\mu) + \Lambda_\mu \partial_\tau \partial_\sigma X^\mu). \end{aligned} \tag{11.33}$$

Thus we see that the bulk terms cancel. Dropping the boundary terms at $\tau = \pm\infty$ gives then the simple, but *non-zero*, result

$$\delta_{\Lambda_\mu} S_B = \int d\tau (\Lambda_\mu \dot{X}^\mu)|_{\sigma=0}^{\sigma=\pi}. \tag{11.34}$$

However, it is only non-zero for directions X^i with Neumann boundary conditions since for D bc we have $X = const$ for all τ. Let us denote these N bc direction as X^m. Then

$$\delta_{\Lambda_\mu} S_B = \int d\tau (\Lambda_m \dot{X}^m|_{\sigma=\pi} - \Lambda_m \dot{X}^m|_{\sigma=0}). \tag{11.35}$$

How can these terms be made to vanish, or canceled?

The crucial insight that comes to the rescue here is that there is a Maxwell field on every D-brane, which is part of a $U(N)$ Yang-Mills field if the D-brane is a brane in a stack. The previous argument for Yang-Mills fields on stacks of D-branes corresponds to the fact that open strings can be considered to have Maxwell-like charges at its ends: $q = +1$ at the $\sigma = \pi$ end and $q = -1$ at the $\sigma = 0$ end. This fact means that we must add to S_B the ordinary EM interaction term (discussed above) at both open string ends:

$$S = S_B + \int d\tau A_m \dot{X}^m|_{\sigma=\pi} - \int d\tau A_m \dot{X}^m|_{\sigma=0}. \tag{11.36}$$

Clearly, if we introduce a new Λ_μ gauge transformation for the Maxwell field the interaction term becomes gauge invariant:

$$\delta_{\Lambda_m} A_m = -\Lambda_m \Rightarrow \delta_{\Lambda_\mu} S = 0. \tag{11.37}$$

But then F_{mn} on the D-brane is no longer gauge invariant but the following combination is

$$\mathcal{F}_{mn} := F_{mn} + B_{mn} \Rightarrow \delta_{\Lambda_m} \mathcal{F}_{mn} = 0. \tag{11.38}$$

On the D-branes the Maxwell theory is now given by the Lagrangian $-\frac{1}{4}\mathcal{F}_{mn}\mathcal{F}^{mn}$ which gives

$$\mathcal{F}_{mn}\mathcal{F}^{mn} = F_{mn}F^{mn} + B_{mn}B^{mn} + 2F_{mn}B^{mn}, \tag{11.39}$$

where the last term indicates that F_{mn} should be interpreted as a string current on the D-brane, i.e., that *the Maxwell field lines (flux) carry string charge that enters the brane from the open string at the point where the string ends.*

D-brane charges:
A final issue in this context is to understand charges associated to the D-branes themselves. This is an interesting question since D-branes, as defined here in terms of open strings, can be argued to appear in the low energy supergravity theories as multi-dimensional generalisations of charged black holes in four-dimensional Einsteinian general relativity. These Reissner-Nordström solutions are point-like solutions of the coupled Einstein-Maxwell equations with both mass and charge.

Thus one might expect stable charged D-branes to exist whenever there is a gauge field they can couple to generalising the two examples we have discussed this far: Maxwell $S_0 = \int_{\mathcal{P}} A$ and $S_1 = -\int_{\Sigma_2} B$ (here p on S_p refers to the number of space dimensions of the D-brane). Note that charges that couple to gauge fields this way are regarded as **electric**. Magnetic charges also occur but these are associated with other D-branes as we will explain below.

First we recall the gauge fields we encountered previously in the context of the RR sector of the type IIA and IIB superstrings, and which Dp-branes they can have an electric coupling to (in the brackets):

$$\text{Type IIA: } A_\mu(D0),\ \ A_{\mu\nu\rho}(D2), \tag{11.40}$$

$$\text{Type IIB: } A(D(-1)),\ \ A_{\mu\nu}(D1),\ \ A^+_{\mu\nu\rho\sigma}(D3). \tag{11.41}$$

These electric coupling terms are (with $X^\mu = X^\mu(\tau, \sigma_1, ..., \sigma_p)$ and the (p+1)-form A_{p+1})

$$S_p = -\int_{\Sigma_{p+1}} A_{p+1} = -\int_{\Sigma_{p+1}} d\tau d\sigma_1...d\sigma_p\, \partial_\tau X^{\mu_1} \partial_{\sigma_1} X^{\mu_2} .... \partial_{\sigma_p} X^{\mu_{p+1}} A_{\mu_1.....\mu_{p+1}}(X) \tag{11.42}$$

Comments:
1) Although the D-branes are new compared to the string discussion above, the two cases $A_\mu(D0)$ and $A_{\mu\nu}(D1)$ are mathematically the same as the charged particle and the string, respectively. Note that the $D1$-brane is not the fundamental string although this latter one is also a solution of the supergravity equations.
2) $A_{\mu\nu\rho}(D2)$ is a direct generalisation with one more space dimension than the string.
3) $A^+_{\mu\nu\rho\sigma}(D3)$ is still one dimension up but the self-duality (the + on the field) makes the D-brane very special. Note that it has four space-time dimensions like our own universe.
4) $A(D(-1))$ is very different from the other cases: Here the D-brane is a *point in space-time*, i.e. what is called an *instanton.*

Before we turn to magnetic charges we need to make clear how the various electric D-brane charges are measured physically. Recall the usual situation in 3+1 Maxwell theory: The electric charge q_e of a point particle is computed using Gauss' law

$$q_e = \frac{1}{4\pi}\int_{S^2} \mathbf{E}\cdot d\mathbf{a} = \frac{1}{4\pi}\int_{S^2} F^{0i} da^i = \frac{1}{4\pi}\int_{S^2} \frac{1}{2}(\star F)^{ij} da^{ij} = \frac{1}{Vol(S^2)}\int_{S^2} \star F. \tag{11.43}$$

Here we have used the general definition of a 2-form

$$F_2 = \frac{1}{2} dx^\mu \wedge dx^\nu F_{\mu\nu}, \tag{11.44}$$

and of the Hodge dual $\star F_2$, which in $D = 4$ dimensions is also a 2-form,

$$\star F_2 = \frac{1}{2} dx^\mu \wedge dx^\nu (\star F_2)_{\mu\nu} = \frac{1}{2} dx^\mu \wedge dx^\nu (\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}). \tag{11.45}$$

Note that the anti-symmetric "wedge" product $dx^\mu \wedge dx^\nu$ is a direct generalisation of the area element $da^i = \epsilon^{ijk} dx^j dx^k$, or $d\mathbf{A} = d\mathbf{a}_1 \times d\mathbf{a}_2$, in three space dimensions.

In the case of a magnetic charge q_m it is computed in a similar way using the magnetic field $\mathbf{B}$:

$$q_m = \frac{1}{4\pi}\int_{S^2} \mathbf{B}\cdot d\mathbf{a} = \frac{1}{4\pi}\int_{S^2} \frac{1}{2}(F)^{ij} da^{ij} = \frac{1}{Vol(S^2)}\int_{S^2} F. \tag{11.46}$$

Thus we see that in $D = 4$ both the electric and magnetic charges are point-like and the surfaces surrounding them is S^2 in both cases.

How are these formulas generalised to the superstring living in 10 spacetime dimensions? The answer is rather clear for the electric charge: $S^2 \to S^8$ which gives

$$q_e^{(D0)} = \frac{1}{Vol(S^8)}\int_{S^8} \mathbf{E}\cdot d\mathbf{a} = \frac{1}{Vol(S^8)}\int_{S^8} \star F_2. \tag{11.47}$$

The surprise is now that the corresponding magnetic charge is not, as in 3+1 dimensions, point-like (i.e., a $D0$ brane) but instead given by (since F_2 is a 2-form)

$$q_m^{(D6)} = \frac{1}{Vol(S^2)}\int_{S^2} \frac{1}{2}F^{ij} da^{ij} = \frac{1}{Vol(S^2)}\int_{S^2} F_2. \tag{11.48}$$

But this implies that the D-brane having this kind of magnetic charge in $D = 10$ spacetime dimensions must be a $D6$-brane. This is the object that can be enclosed by a two-sphere S^2 in nine space dimensions. However, we have not yet mentioned Dp-branes with this high value of p but they are known to exist as solutions to the relevant supergravity theory in $D = 10$. In fact, each of the electric branes mentioned above has a **dual brane** with a magnetic charge measured by the same gauge field strength: *If Dp has electric charge then the dual brane having a magnetic charge is the $D(6-p)$-brane.* Now it is clear what makes the $D3$-brane so special: It is self-dual in the sense that it can itself have both electric and magnetic charges (like points in three space dimensions).

Comment: In M-theory the low-energy supergravity theory in $D = 11$ has two solutions of space dimensions 2 and 5, the so called $M2$-brane and its dual the $M5$-brane. They couple both (as above) to the M-theory field $A_{\mu\nu\rho}$, $M2$ electrically and $M5$ magnetically.

Comment: One can carry out the above discussion about charges in terms of the dual field strength $\tilde{F} := \star F$ instead. In that case the only thing that happens is that the electric and magnetic nature of the charges flips.

Note: The only two formulas used in the above discussion of integrations over spheres are the p**-form**

$$F_p := \frac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge .... \wedge dx^{\mu_p} F_{\mu_1 ... \mu_p}, \tag{11.49}$$

where $dx^{\mu_1} \wedge dx^{\mu_2} \wedge .... \wedge dx^{\mu_p}$ is the p-dimensional volume element, and the **dual** p**-form** which is $D-p$-form:

$$(\star F)_{\mu_1 ... \mu_{D-p}} := \frac{1}{p!}\epsilon_{\mu_1 ... \mu_{D-p}}{}^{\mu_{D-p+1}...\mu_D} F_{\mu_{D-p+1}...\mu_D}, \tag{11.50}$$

The relation of a D-form to the volume element $d^D x \sqrt{|\det g|}$ in a generally curved d-dimensional manifold is given in terms of the Levi-Civita symbol ϵ (with $\epsilon^{01....D} = +1$) by

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge .... \wedge dx^{\mu_D} = d^D x \ \epsilon^{\mu_1 ... \mu_D} = d^D x \sqrt{|\det g|}\, \tilde{\epsilon}^{\mu_1 ... \mu_D} . \tag{11.51}$$

Note: The LHS is an antisymmetric *tensor*, not a tensor density, as is also the case for $\tilde{\epsilon}$.

Examples: In $d = 3$ the magnetic field $B^i = \frac{1}{2}\epsilon^{ijk} F^{jk}$ and in $D = 4$ the duality $(\star F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}$. Note that there $\star^2$ can give a minus sign (depending on signature and dimensionality D). For examples in 3+1 dimensions $\star^2 = -1$. This implies that splitting $F_{\mu\nu}$ into self-dual and anti-self-dual parts makes these objects complex. In $D = 6$ we instead have $\star^2 = +1$. This last fact is important for the theories living on some 5-branes, the $M5$ in M-theory and the magnetic dual of the fundamental string, the so called $NS5$-brane.

12 Lecture 14

In this lecture we will discuss how the string depends on the radius of a compactified dimension. The result, which is rather surprising, is known as **T-duality**. Since it is easier to derive these effects for closed strings than for the open ones we start with closed strings. Before this, however, we make a few comments on *duality in field theory.*

12.1 Chapter 17: T-duality for closed strings

Duality: This is a very deep and important concept and refers to the possibility to describe the same physics in terms two different theories[37]. Often the field theories are of the same kind but with interesting relations between the parameters of the theories. In particular, this is true when the coupling constant is inverted, called **S-duality** or when the radius of a compact dimension is inverted, called **T-duality**. Apart from these parameter based duality transformations between the two descriptions there are some very special cases where the field theories themselves are totally different and may even live in different dimensions. Some examples are:
1) The duality between 11-dimensional supergravity and type IIA supergravity in 10 dimensions (S-duality) and between type IIA and IIB (T-duality).
2) The duality between supergravity in D-dimensional AdS and a non-gravitational conformal field theory (CFT) on the boundary of AdS_D. If the CFT is strongly coupled, and hence very hard to deal with computationally, the dual gravity theory in AdS is weakly coupled and easy to deal with. This phenomenon is known as the **AdS/CFT correspondence**. It has found an enormous number of applications in non-gravitational physics, ranging from QCD to superconductivity.
3) Duality is also related to some celebrated areas in mathematics, e.g., the theory of automorphic forms and the so called **Langlands program**.

A simple example of these ideas appears already in Maxwell's equations without sources (i.e., when $j^\mu = 0$):

$$\nabla \cdot \mathbf{E} = 0,\ \ \nabla \times \mathbf{B} = \frac{1}{c}\partial_t \mathbf{E},\ \ \nabla \cdot \mathbf{B} = 0,\ \ \nabla \times \mathbf{E} = -\frac{1}{c}\partial_t \mathbf{B}. \tag{12.1}$$

Clearly, these equations are left invariant by the following duality transformation:

$$(\mathbf{E}, \mathbf{B}) \to (-\mathbf{B}, \mathbf{E}). \tag{12.2}$$

One may note that while the Hamiltonian $\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$ is invariant the Lagrangian is not since $\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \to -\mathcal{L}$.

Let us now turn to the closed bosonic string moving in a spacetime with the X^{25} compactified on a circle, i.e., $X^{25} \in S^1_R$. Thus, if the radius is R we have $X^{25} \in [0, 2\pi R]$.

[37] See "Duality of fields and strings" by Joe Polchinski, hep-th/1412.5704.

On this spacetime there are two kinds of closed strings: (view these two cases, e.g., on a cylinder)

1) **Untwisted sector**: Closed strings that do not wind around the circle S^1_R.
2) **Twisted sector**: Closed strings that do wind around the circle S^1_R.

This implies that:

1) Untwisted sector: All components of X^μ have the standard mode expansion derived previously in a uncompactified spacetime.
2) Twisted sector: All X^μ except X^{25} have the same mode expansion as in 1), while X^{25} satisfies (view this on an infinite x^{25} coordinate with the equivalence relation $x^{25} \sim x^{25} + 2\pi R$)

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + m(2\pi R), \;\; m \neq 0, \;\; (m \in \mathbf{Z}). \tag{12.3}$$

This new integer m is called the **winding number**.

Before deriving the mode expansion of the quantum operator $X^{25}(\tau, \sigma)$, it is convenient to introduce the **winding operator** w with eigenvalues $\frac{mR}{\alpha'}$. That is, we now have

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi\alpha' \, w. \tag{12.4}$$

The final answer for the mode expansion in the twisted sector is rather interesting so let us derive it in detail. As usual for the closed string solving the wave equation gives rise to two independent functions (for left and right moving wave modes)

$$X(\tau, \sigma) = X_L(u) + X_R(v), \;\; u = \tau + \sigma, \;\; v = \tau - \sigma. \tag{12.5}$$

Then the definition of the twisted sector field above implies

$$X_L(u + 2\pi) - X_L(u) = X_R(v) - X_R(v - 2\pi) + 2\pi\alpha' \, w. \tag{12.6}$$

Taking u and v derivatives of this equation implies that $X'_L(u)$ and $X'_R(v)$ are 2π periodic functions, that is, we get the same expansions as for the untwisted case

$$X_L(u) = x_{0,L} + \sqrt{\frac{\alpha'}{2}} \, \bar{\alpha}_0 \, u + i\sqrt{\frac{\alpha'}{2}} \, \Sigma_{n \neq 0} \frac{1}{n} \bar{\alpha}_n e^{-inu}, \tag{12.7}$$

$$X_R(v) = x_{0,R} + \sqrt{\frac{\alpha'}{2}} \, \alpha_0 \, v + i\sqrt{\frac{\alpha'}{2}} \, \Sigma_{n \neq 0} \frac{1}{n} \alpha_n e^{-inv}, \tag{12.8}$$

Inserting these expansions into the winding condition above gives

$$\bar{\alpha}_0 - \alpha_0 = \sqrt{2\alpha'} \, w \Rightarrow w = \frac{1}{\sqrt{2\alpha'}} (\bar{\alpha}_0 - \alpha_0). \tag{12.9}$$

So together with the result for the momentum mode

$$p = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, (\dot{X}_L + \dot{X}_R) = \frac{1}{\sqrt{2\alpha'}} \, (\bar{\alpha}_0 + \alpha_0), \tag{12.10}$$

we now have two momentum like operators, p and w, and hence we must have two coordinates dual to these: $x_{0,L}$ and $x_{0,R}$.

The way to organise the zero mode sector for the compactified closed string coordinate is as follows: In analogy with (from above)

$$\bar{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}(p+w), \ \ \alpha_0 = \sqrt{\frac{\alpha'}{2}}(p-w), \tag{12.11}$$

we introduce a new operator q which is the canonical coordinate of w, i.e., $[q,w]=i$. Note that we already have $[x_0,p]=i$ where $x_0 = x_{0,L}+x_{0,R}$ and $p = \frac{1}{\sqrt{2\alpha'}}(\bar{\alpha}_0+\alpha_0)$. Then it follows that $q := x_{0,L}-x_{0,R}$ and we have

$$x_{0,L} = \frac{1}{2}(x_0+q), \ \ x_{0,R} = \frac{1}{2}(x_0-q). \tag{12.12}$$

Thus we can write the two now *completely independent* left and right moving expansions for the compactified closed string above as follows:

$$X_L(u) = x_{0,L} + \frac{\alpha'}{2}\,p_L\,u + i\sqrt{\frac{\alpha'}{2}}\,\Sigma_{n\neq 0}\frac{1}{n}\bar{\alpha}_n e^{-inu}, \tag{12.13}$$

$$X_R(v) = x_{0,R} + \frac{\alpha'}{2}\,p_R\,v + i\sqrt{\frac{\alpha'}{2}}\,\Sigma_{n\neq 0}\frac{1}{n}\alpha_n e^{-inv}, \tag{12.14}$$

where we have defined the left and right center of mass momenta

$$p_L := p+w, \ \ \ p_R := p-w. \tag{12.15}$$

Propagators and vertex operators:
After a Wick rotation we get a holomorphic $X_R(z)$ and an antiholomorphic $X_L(\bar{z})$:

$$X_R(z) = x_{0,R} + \frac{\alpha'}{2}\,p_R\,\ln z + i\sqrt{\frac{\alpha'}{2}}\,\Sigma_{n\neq 0}\frac{1}{n}\alpha_n z^{-n} \tag{12.16}$$

$$X_L(\bar{z}) = x_{0,L} + \frac{\alpha'}{2}\,p_L\,\ln \bar{z} + i\sqrt{\frac{\alpha'}{2}}\,\Sigma_{n\neq 0}\frac{1}{n}\bar{\alpha}_n \bar{z}^{-n}, \tag{12.17}$$

Clearly, in this case, we have two independent propagators also:

$${}_x\langle 0|X_R(z)X_R(w)|0\rangle_p = -\frac{\alpha'}{2}\ln(z-w), \ \ {}_x\langle 0|X_L(\bar{z})X_L(\bar{w})|0\rangle_p = -\frac{\alpha'}{2}\ln(\bar{z}-\bar{w}). \tag{12.18}$$

where we have used

$$[x_{0,L},p_L]=i, \ \ [x_{0,R},p_R]=i. \tag{12.19}$$

This is good point to introduce the concept of **vertex operator**. These can be used to compute interactions and scattering amplitudes in the low energy field theory associated to any string. Let us define the normal ordered exponential of the compactified string coordinate defined above (we use the R sector but suppress the index R, and use kX for a Lorentzian scalar product)

$$V_k(z) =: e^{ikX(z)} := e^{ikX^-(z)} e^{ikX^+(z)}, \text{ satisfying } {}_x\langle 0|V_k(z)|0\rangle_p = 1. \tag{12.20}$$

Here we have used the following definitions (just as in QFT)

$$X^+(z) := \frac{\alpha'}{2}\, p \ln z + i\sqrt{\frac{\alpha'}{2}}\, \Sigma_{n\geq 1} \frac{1}{n} \alpha_n z^{-n} \Rightarrow X^+(z)|0\rangle_p = 0 \tag{12.21}$$

$$X^-(z) := x_0 + i\sqrt{\frac{\alpha'}{2}}\, \Sigma_{n\leq -1} \frac{1}{n} \alpha_n z^{-n} \Rightarrow {}_x\langle 0|X^-(z) = 0. \tag{12.22}$$

Using the Baker-Hausdorff formula it is then rather simple to compute the product of two such vertex operators located at two different points on the Euclidean world-sheet. We get

$$V_{k_1}(z_1) V_{k_2}(z_2) =: V_{k_1}(z_1) V_{k_2}(z_2) : e^{-[k_1 X^+(z_1), k_2 X^-(z_2)]}, \tag{12.23}$$

where to get over-all normal ordering we had to flip the order of the two exponential factors that were not normal ordered in the original product:

$$e^{ik_1 X^+(z_1)} e^{ik_2 X^-(z_2)} = e^{[ik_1 X^+(z_1), ik_2 X^-(z_2)]} e^{ik_2 X^-(z_2)} e^{ik_1 X^+(z_1)}. \tag{12.24}$$

valid if the commutator in the exponent is a c-number (i.e., not an operator).

Then using the propagator result above, which is equal to the commutator in the exponent, we see that

$$e^{ik_1 X^+(z_1)} e^{ik_2 X^-(z_2)} = (z_1 - z_2)^{\frac{\alpha'}{2} k_1 \cdot k_2} e^{ik_2 X^-(z_2)} e^{ik_1 X^+(z_1)}. \tag{12.25}$$

Thus the final answer for the operator product is

$$V_{k_1}(z_1) V_{k_2}(z_2) = (z_1 - z_2)^{\frac{\alpha'}{2} k_1 \cdot k_2} : V_{k_1}(z_1) V_{k_2}(z_2) : . \tag{12.26}$$

Comment:

It is an interesting fact that setting $\alpha' = 2$ and letting in one dimension (no μ indices) the momenta be $\pm\sqrt{2}$, the two vertex operators above together with the operator $\partial_z X(z)$ generate the $SU(2)$ Kac-Moody algebra which is an infinite dimensional version of the ordinary $SU(2)$ Lie algebra (with the three generators $V_0, V_\pm$)[38]. This kind of Kac-Moody algebras are quite similar to the Virasoro algebra which, however, is not related to any Lie algebra in the same way. In the heterotic string the gauge groups $E_8 \times E_8$ and $SO(32)$ are constructed in the same (from their root lattices).

[38]If you have studied Lie algebra representation theory you may recognise the momenta $\pm\sqrt{2}$ as the root lattice vectors of the Lie algebra $SU(2)$. The full Kac-Moody algebra is obtained using vertex operator techniques and mode expansions of the vertex operators $V_{\pm\sqrt{2}}(z) = \Sigma_{n\in\mathbf{Z}} V_n^{\pm} z^{-n}$ and "current" $\partial_z X(z) = \Sigma_{n\in\mathbf{Z}} H_n z^{-n}$. These operators give the *level* $k = 1$ $SU(2)$ Kac-Moody algebra which general level k reads

$$[V_n^+, V_m^-] = \sqrt{2} H_{n+m} + kn\delta_{n+m,0}, \quad [H_n, V_m^{\pm}] = \pm\sqrt{2} V_{n+m}^{\pm}, \quad [H_n, H_m] = kn\delta_{n+m,0}. \tag{12.27}$$

Note that the zero mode (n=m=0) subalgebra is just the ordinary $SU(2)$ Lie algebra.

T-duality:
We should now try to understand the physical consequences of the structure of the zero modes discussed above in the compactified closed string case. First we need the transverse $n = 0$ Virasoro generator, using the right moving sector and the index split $I = (i, 25)$,

$$L_0^\perp = \frac{1}{2}\alpha_0^I\alpha_0^I + \Sigma_{n=1}^\infty \alpha_{-n}^I\alpha_n^I = \frac{1}{2}\alpha_0^i\alpha_0^i + \frac{1}{2}\alpha_0^{25}\alpha_0^{25} + \Sigma_{n=1}^\infty \alpha_{-n}^I\alpha_n^I. \tag{12.28}$$

Then using the ordinary relation between α_0 and momenta, that is $\alpha_0^i = \sqrt{\frac{\alpha'}{2}}p^i$ but in the compactified direction $\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}(p - w)$, we find that

$$L_0^\perp = \frac{\alpha'}{4}p^ip^i + \frac{\alpha'}{4}(p - w)^2 + N^\perp. \tag{12.29}$$

Similarly for the left moving sector we get

$$\bar{L}_0^\perp = \frac{\alpha'}{4}p^ip^i + \frac{\alpha'}{4}(p + w)^2 + \bar{N}^\perp. \tag{12.30}$$

The level matching condition $L_0^\perp = \bar{L}_0^\perp$ picks up new terms from the compactified direction:

$$L_0^\perp = \bar{L}_0^\perp \Rightarrow \alpha' pw = N^\perp - \bar{N}^\perp. \tag{12.31}$$

Also the mass-square in the **uncompactified** directions will be affected by the compact direction:

$$M^2 = 2p^+p^- - p^ip^i = \frac{1}{2}(p + w)^2 + \frac{1}{2}(p - w)^2 + \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2), \tag{12.32}$$

where the momenta in direction 25 has been moved over to the RHS. Combining these momentum terms in M^2 we get

$$M^2 = p^2 + w^2 + \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2), \tag{12.33}$$

The last step before we can discuss the physical implications is to replace p and w in the level matching condition and mass formula by their eigenvalues $p = n\frac{1}{R}$ and $w = m\frac{R}{\alpha'}$ where both m and n are integers:

$$M^2 = \frac{n^2}{R^2} + \frac{m^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2), \tag{12.34}$$

$$N^\perp - \bar{N}^\perp = m\,n. \tag{12.35}$$

State space in target spacetime with one S^1 direction
In the following discussion the uncompactified part of spacetime is the part of interest to us so we will set $D = 1 + 24$ and use indices $\mu, \nu, ... = 0, 1, 2, ..., 24$. The index value 25 is suppressed from now on (as done already in the last formulas above).

The ground state, i.e., the lowest level state, is obtained for $(N, \bar{N}) = (0,0)$ and $(n,m) = (0,0)$ where (n,m) are the eigenvalues of (p,w) in the compact direction:

$$|p^\mu;(0,0)\rangle, \quad M^2 = -\frac{4}{\alpha'}, \quad T(x): \text{ tachyon in 25 spacetime dimensions.} \tag{12.36}$$

The next level is a bit more interesting since it contains different kinds of states. This happens since we can satisfy the new level matching condition in more than one way and there are different kinds of oscillators:

1) $N^\perp = \bar{N}^\perp = 1$ and $n = m = 0$ gives $M^2 = 0$ (with either n or m non-zero we get massive states)

$$\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|p^\mu;(0,0)\rangle, \ \alpha^\mu_{-1}\bar{\alpha}_{-1}|p^\mu;(0,0)\rangle, \ \alpha_{-1}\bar{\alpha}^\nu_{-1}|p^\mu;(0,0)\rangle, \ \alpha_{-1}\bar{\alpha}_{-1}|p^\mu;(0,0)\rangle, \tag{12.37}$$

corresponding to the massless metric, Kalb-Ramond and dilaton fields plus two massless vector gauge fields and another scalar field. This set of fields is obtained by a direct field theory compactification of the massless fields in $D = 26$ dimensions, that is, apart from the dilaton, the metric and Kalb-Ramond field.

2) If $n = \pm 1$ AND $m = \pm 1$ then level-matching requires either $(N^\perp, \bar{N}^\perp) = (1,0)$ or $(N^\perp, \bar{N}^\perp) = (0,1)$ in which case we get $M^2 = 0$ for special values of the radius R. This part of the spectrum is a pure string phenomenon and cannot be understood in terms of the low-energy field theory:

$$\alpha^\mu_{-1}|p^\mu;(\pm,\pm)\rangle, \ \alpha_{-1}|p^\mu;(\pm,\pm)\rangle, \tag{12.38}$$

$$\bar{\alpha}^\mu_{-1}|p^\mu;(\pm,\mp)\rangle, \ \bar{\alpha}_{-1}|p^\mu;(\pm,\mp)\rangle. \tag{12.39}$$

In all these cases the mass square is

$$M^2 = \frac{1}{R^2} + \frac{R^2}{\alpha'^2} - \frac{2}{\alpha'} = (\frac{1}{R} - \frac{R}{\alpha'})^2. \tag{12.40}$$

This purely stringy result is very interesting since it implies that at the special value of the radius $R = \sqrt{\alpha'}$ the states are massless, and for all other values they are massive.

Thus for the particular value of the radius $R = \sqrt{\alpha'}$ we have four massless vector fields which can be shown to generate an interacting Yang-Mills theory with gauge group $SU(2) \times U(1)$. The trick the string is using to get this amazing result is to make use of the $SU(2)$ vertex operators and their Kac-Moody algebra we mentioned above[39]. This phenomenon that appears only in string theory and only for the radius $R = \sqrt{\alpha'}$ is called **symmetry enhancement**. This technique to generate Yang-Mills interactions in a closed string theory

[39] In general the Cartan subalgebra of the gauge group can be understood in the low-energy field theory while the generators of the Lie algebra corresponding to roots (i.e., the step operators) are represented by vertex operators.

is also the basis of the **heterotic string** construction of the two possible gauge groups $E_8 \times E_8$ and $SO(32)/\mathbf{Z}_2$ for which the Cartan subalgebra is 16-dimensional (note that 16 = 26 -10).

T-duality: Finally we can address another (perhaps well-known) string phenomenon, namely what happens when the radius R is inverted. Let us first define the T-duality transformation and observe its effect on the mass spectrum of the closed string

$$\text{T-duality: } R \to \tilde{R} := \frac{\alpha'}{R}, \text{ and } m \leftrightarrow n \Rightarrow M^2(R;(n,m)) = M^2(\tilde{R};(m,n)). \tag{12.41}$$

Since the spectrum as a whole is left invariant we can regard T-duality as a symmetry transformation.

We should now recall the origin of R, namely that it is actually an expectation value of the $D = 24$ scalar field $g_{25,25}$. This field has no potential in $D = 24$ which means that any value of R is possible and thus plays the role of a parameter in the $D = 24$ theory: Such a parameter is called a **modulus**. In more complicated compactifications there may be many scalars like this which then parametrise a space called **moduli space**. These spaces play are crucial role in string theory and are also important objects in mathematics.

The physical implications of the facts above is that the string theory associated with a given value of $R \leq \sqrt{\alpha'}$ is by T-duality equivalent to a string theory with $\tilde{R} \geq \sqrt{\alpha'}$. In this sense there is a minimal distance given by $\tilde{R} = \sqrt{\alpha'}$.

There is another way to express the T-duality by noting that $m \leftrightarrow n$ means that $p \leftrightarrow w$. Then we see from the mode expansions of $X_L(u)$ and $X_R(v)$ that the momentum terms behaves as $p_L = p + w \to +p_L$ but $p_R = p - w \to -p_R$. Extending this to the whole mode expansions we may define T-duality as

$$X_L \to +X_L, \quad X_R \to -X_R, \text{ or as} \tag{12.42}$$

$$X \to \tilde{X}, \text{ where } X = X_L + X_R, \ \tilde{X} := X_L - X_R. \tag{12.43}$$

Note that the T-duality transformation is a symmetry of the whole theory since the string Hamiltonian is invariant and that this symmetry statement can be shown to be true also at the interacting level (using vertex operators).

12.2 Chapter 18: T-duality for open strings

The previous discussion of T-duality for the closed string led to some interesting results about minimal distances and moduli spaces. When this concept is taken over to the open string some new phenomena appear related to D-branes.

Recall the mode expansion we found in the previous discussion of a closed string with a component taking values on the circle of a compactified dimension

$$X(\tau,\sigma) = X_L(\tau+\sigma) + X_R(\tau-\sigma), \tag{12.44}$$

where the two completely independent parts are

$$X_L(u) = \frac{1}{2}(x_0+q) + \frac{\alpha'}{2}(p+w)(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\,\Sigma_{n\neq 0}\frac{1}{n}\bar{\alpha}_n e^{-in(\tau+\sigma)}, \tag{12.45}$$

$$X_R(v) = \frac{1}{2}(x_0-q) + \frac{\alpha'}{2}(p-w)(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\,\Sigma_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau-\sigma)}. \tag{12.46}$$

The eigenvalues of the momentum operators are

$$p = \frac{n}{R},\;\; w = \frac{mR}{\alpha'},\;\; n,m \in \mathbf{Z}. \tag{12.47}$$

The spectrum of a closed string in a target space with one such compact dimension is determined by

$$M^2 = (\frac{n}{R})^2 + (\frac{mR}{\alpha'})^2 + \frac{2}{\alpha'}(N+\bar{N}-2), \tag{12.48}$$

$$N - \bar{N} = n\cdot m. \tag{12.49}$$

This spectrum is T-duality invariant under the transformation

$$R \to \tilde{R} := \frac{\alpha'}{R},\;\; n \leftrightarrow m. \tag{12.50}$$

We note that this means that $p \to w$ and $w \to p$, or

$$p_L = p + w \to p_L,\;\; p_R = p - w \to -p_R. \tag{12.51}$$

Viewing these last two relations as part of the mode expansions it suggests that this result should be extended to the whole mode expansions[40], i.e.,

$$X_L \to X_L,\;\; X_R \to -X_R. \tag{12.52}$$

Finally, we introduce a new string coordinate along with the usual one $X = X_L + X_R$, namely

$$\tilde{X} := X_L - X_R \tag{12.53}$$

Now we use the results above for the compactified closed string to analyse the compactified

[40] This conclusion is important when checking T-duality when interactions are considered.

open string with (N,N) bc on all 26 components X^μ. This open string we view as having its ends attached to a *space-filling* D25-brane (thus filling up all 25+1 spacetime dimensions). Such a string cannot have any winding modes in the mode expansion of any component, including the compact 25th space dimension with $X^{25} \in S^1_R$. This is because the ends are free to move and any winding can therefore be undone. This situation is standard in all uncompactified dimensions so we discuss now only X^{25}, again dropping the 25. The mode expansion for this open string X is obtained from the closed string mode expansion above for $X = X_L + X_R$ by setting all $\bar{\alpha}_n = \alpha_n$ for all $n \in \mathbf{Z}$. This gives

$$X = X_L + X_R = x_0 + 2\alpha'\, p\,\tau + i\sqrt{2\alpha'}\Sigma_{n\neq 0}\frac{1}{n}\alpha_n\, e^{-in\tau}\cos n\sigma, \tag{12.54}$$

which is the same mode expansion as the one derived before for an open string with (N,N) bc. In particular we find the $\cos n\sigma$ characteristic of these boundary conditions and the zero modes operators x_0, p where the momentum has discrete eigenvalues $p = \frac{n}{R}$. The N bc can be written

$$\partial_\sigma X| = \partial_\sigma(X_L + X_R)| = 0 \Rightarrow (X'_L - X'_R)| = 0. \tag{12.55}$$

We now come to the key point here. Performing a T-duality transformation we get

$$X = X_L + X_R \to \tilde{X} = X_L - X_R = q_0 + 2\alpha'\, p\,\sigma + \sqrt{2\alpha'}\Sigma_{n\neq 0}\frac{1}{n}\alpha_n\, e^{-in\tau}\sin n\sigma. \tag{12.56}$$

Strangely enough, the T-duality transformation turns the (N,N) open string into a (D,D) open string which is seen by the sin function in the oscillator part of the mode expansion. The zero modes are now q_0, p which are not operators (note that p multiplies σ not τ) since the (D,D) open string expansion is obtained by identification $q_0 := \bar{x}_1$ and $2\alpha' p := \frac{1}{\pi}(\bar{x}_2 - \bar{x}_1)$. The conclusion is then that a T-duality transformation transforms a D bc into a N bc and vice versa. In the case studied here the $D25$ brane is transformed into a $D24$ brane by the T-duality transformation.

Note also that the open string after T-duality from $X \in S^1_R$ to $\tilde{X} \in S^1_{\tilde{R}}$ gives

$$\tilde{X}(\tau, \sigma = \pi) - \tilde{X}(\tau, \sigma = 0) = 2\pi\,\alpha'\, p = 2\pi\,\alpha'\,\frac{n}{R} = 2\pi\tilde{R}n. \tag{12.57}$$

This open string has its two ends fixed at the D-brane having a fixed position on the $S^1_{\tilde{R}}$ and can therefore have winding, here given by the integer n in the above formula.

The perhaps simplest way to see the change of boundary conditions is as follows

$$\text{N bc}: \ \partial_\sigma X| = \partial_\sigma(X_L + X_R)| = (X'_L - X'_R)| = 0 \to (\tilde{X}'_L + \tilde{X}'_R)| = \partial_\tau \tilde{X}| = 0: \ \text{D bc}. \tag{12.58}$$

13 Lecture 15

A nice way to approach string theory if we want to understand the physics content of it is to derive its *low energy effective field theory*. This can be viewed as an approximation obtained by letting the energy (in, e.g., some physical scattering amplitude) get smaller and smaller whereby the string shrinks (due to tension) and starts behaving like a field theory based on point particle excitations, i.e., it looks more and more like a QFT. The natural length scale to relate such limits to is the Planck energy.

In this lecture we will develop some insight into the nature of the *low energy effective field theories* of the open string massless fields, which we have seen live on the D-brane. We will be concentrating on the vector gauge field $A_\mu(x)$ where x are coordinates **on the D-brane**. After an initial discussion of boundary conditions and moving D-branes, we turn to non-linear aspects which are of two kinds: 1) Non-linear Maxwell fields, i.e., Born-Infeld theory, and 2) Yang-Mills fields from intersecting stacks of D-branes, which is the topic of the next lecture where the Standard Model is (partly) extracted from string theory.

We start with a general discussion of the action functional of the effective field theory which will be denoted as

$$S^{eff}[g_{\mu\nu}, ..., A_\mu, ...; \alpha', g_s, ...], \tag{13.1}$$

where we have added the dependence of the theory on the parameters, g_s and α', as well as other possible ones called **moduli**. These latter ones are VEVs of the (many) scalar fields appearing in the string theory, e.g., various radii after compactification. As for the fields we note that there are terms in S^{eff} which are integrals over all ten dimensions (we will have in mind superstrings in the following discussions), as in the Einstein-Hilbert term, but also integrals over the dimensions of various D-branes where the vector fields live. Note that the vector fields that arise from the 10D metric when compactifying are of a different nature.

As discussed previously, the string loop-expansion is associated with the Euler number $\chi = 2-2g$ where g is the *genus* of a closed Riemann surface related to a given closed string loop-diagram (a "plumbing" diagram). When we include D-branes into this discussion the open strings are related to Riemann surfaces with boundaries and the Euler number changes to[41]

$$\chi = 2 - 2g - b, \;\; \text{b is the number of disconnected boundaries.} \tag{13.2}$$

This implies that while the Einstein-Hilbert term comes with a factor $1/g_s^2$ as we have argued for in a previous lecture, D-brane terms are related to Riemann surfaces with one boundary and hence come with a factor $1/g_s$ (at tree level). There is another kind of branes in string theory coming from the (NS, NS) sector which are dual to the fundamental string

[41]Is it possible to add yet another term, the number of *cross caps*, relevant for closed strings without orientation on the world-sheet. E.g., the sphere with one cross cap is a Möbius strip and with two cross caps it becomes a Klein bottle.

(they have a magnetic charge under the $B_{\mu\nu}$ field, recall the previous discussion of electric and magnetic D-brane charges). These $NS5$-branes come instead with a factor $1/g_s^2$ which, however, cannot be explained by this new Euler number since these branes are not related to open strings.

One important feature of this action functional is that it is a dubbel sum

$$S^{eff}[g_{\mu\nu}, ..., A_\mu, ...; \alpha', g_s, ...] = \Sigma_n (\alpha')^n \Sigma_\chi \, g_s^{-\chi} S^{eff}_{\chi,n}. \tag{13.3}$$

As mentioned previously these terms can in principle be computed using 1) vertex operators or 2) world-sheet beta-functions. However, due to the enormously complicated structure of the higher derivative terms only a few of them are known. Note that the counterterm expansion in ordinary gravity theories (with no more than two derivatives in the classical Lagrangian) is a power series in Newton's constant $G_N^{(10)} = g_s^2(\alpha')^4$.

In the context of D-branes and their effective low-energy field theories we will in this lecture, however, use another method to argue for the non-linear version of Maxwell theory that the string generates. Thus we are seeking $S^{eff}_{D-brane}[A_\mu]$ to lowest order in g_s and see what this means for the α' dependence.

Comment: The higher derivative terms in the effective action of course means that the theory is not power-counting renormalisable, not even if we neglect all the gravitational interactions (see below). However, this is not a problem since string theory is **finite**, i.e., all string loop integrals are finite[42] at both UV[43] and IR[44] limits of the loop-momenta. This is equivalent to the fact that the coefficients in the gravity counterterm expansion, which need experiments to be determined in Einstein's gravity theory, are all determined in string theory (but very hard to compute).

13.1 Chapter 19: Electromagnetic fields on D-branes

We start by recalling the action describing how the open string couples to a Dp-brane: (with x^m $(m = 0, 1, ..., p = (0, i))$ being coordinates on the Dp-brane)

$$S[X] = \int d\tau d\sigma \mathcal{L}_{NG}(\dot{X}, X') + \int d\tau \, A_m(X)\partial_\tau X^m|_{\sigma=\pi} - \int d\tau \, A_m(X)\partial_\tau X^m|_{\sigma=0}. \tag{13.4}$$

We can interpret this by associating to each end of the string an electric charge that couples to the Maxwell vector potential living on the D-brane where the open string ends.

[42] See Ed Witten, "Perturbative Superstring Theory Revisited" (see end of Introduction), hep-th/1209.5461.

[43] This is due to the modular invariance of loop-amplitudes, see e.g. Shapiro Phys. Rev. **D5** (1972) p. 1945-8 and the books by Green, Schwarz and Witten.

[44] See the penultimate footnote.

Consider now the simplest possible field strength of A_m, namely that with a constant F_{mn}. As usual we have $F_{0i} = -E_i$ and $F_{ij} = B_{ij}$. As is well-known from previous courses one can choose a gauge where the vector potential is simply linear in the coordinates

$$A_n = \frac{1}{2} x^m F_{mn}. \tag{13.5}$$

Then we insert this vector potential into the action above and perform the usual steps to find the boundary conditions from Hamilton's principle. We get, indicating by $S[X; F_{mn}]$ the dependence on the constant "parameters" F_{mn},

$$S[X; F_{mn}] = \int d\tau d\sigma \mathcal{L}_{NG}(\dot{X}, X') + \frac{1}{2}\int d\tau\, F_{mn} X^m \dot{X}^n|_{\sigma=\pi} - \frac{1}{2}\int d\tau\, F_{mn} X^m \dot{X}^n|_{\sigma=0}. \tag{13.6}$$

The vanishing of the variation of the action under δX^μ (i.e., in all μ directions) gives

$$\delta S = \int d\tau d\sigma (\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\delta \dot{X}^\mu + \frac{\partial \mathcal{L}}{\partial X'^\mu}\delta X'^\mu) \tag{13.7}$$

$$+ \frac{1}{2}\int d\tau\, F_{mn}\delta X^m \dot{X}^n\Big|_{\sigma=0}^{\sigma=\pi} + \frac{1}{2}\int d\tau\, F_{mn} X^m \delta\dot{X}^n\Big|_{\sigma=0}^{\sigma=\pi} = 0. \tag{13.8}$$

As usual we must now integrate by parts to move the derivatives away from the variations (in the first line and 2nd term in the 2nd line). This gives, dropping the boundary terms in the τ direction)

$$\delta S = -\int d\tau d\sigma \left(\partial_\tau(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}) + \partial_\sigma(\frac{\partial \mathcal{L}}{\partial X'^\mu})\right)\delta X^\mu \tag{13.9}$$

$$+ \int d\tau \frac{\partial \mathcal{L}}{\partial X'^\mu}\delta X^\mu\Big|_{\sigma=0}^{\sigma=\pi} + \int d\tau\, F_{mn}\delta X^m \dot{X}^n\Big|_{\sigma=0}^{\sigma=\pi} = 0. \tag{13.10}$$

So, $\delta S = 0$ implies the usual string world-sheet field equation from the bulk term above (1st line) and from the 2nd line we get the usual boundary result for the directions orthogonal to the D-brane ($\mu \neq m$) but a new kind of boundary condition in the D-brane directions ($\mu = m$)

$$\frac{\partial \mathcal{L}}{\partial X'^m} + F_{mn}\dot{X}^n = 0 \text{ at } \sigma = 0, \pi. \tag{13.11}$$

or, after gauge fixing,

$$X'_m - 2\pi\alpha' F_{mn}\dot{X}^n = 0 \text{ at } \sigma = 0, \pi. \tag{13.12}$$

This result means that the usual N bc now, due to the vector field living on the D-brane, is mixed up with the D bc (sometimes called Robin boundary conditions).

Let us now try to see what the physics of these new boundary conditions is. Consider first the **magnetic** case ($m = (0, i)$)

$$F_{mn}: \quad F_{0i} = 0, \;\; F_{ij} \neq 0 \Rightarrow X'^0 = 0, \;\; X'^i \neq 0 \text{ at } \sigma = 0, \pi. \tag{13.13}$$

As an example we set $F_{23} = B$. Then

$$X'^2 = 2\pi\alpha' B\, \dot{X}^3, \;\; X'^3 = -2\pi\alpha' B\, \dot{X}^2 \text{ at } \sigma = 0, \pi, \tag{13.14}$$

which tell us that for $B = 0$ both X^2 and X^3 have N bc and for $B = \infty$ they both have D bc, and for other values of B they satisfy Robin bc. Note that for $B = \infty$ the D b.c. seems to say that there are $D(p-2)$-branes oriented orthogonally to the X^2 and X^3 directions.

Let us now discuss the electric case: $F_{0i} \neq 0$ for one direction, $i = 25$ say, and $F_{ij} = 0$. Then

$$X_0' = -2\pi\alpha'\, F_{0,25}\dot{X}^{25} = 0, \;\; X_{25}' = -2\pi\alpha'\, F_{25,0}\dot{X}^0 = 0 \;\text{ at }\; \sigma = 0, \pi. \tag{13.15}$$

Setting $\mathcal{E} = 2\pi\alpha'\, F_{25,0}$ we can write these equations as ($X^{25} := X$ and $X_0 = -X^0$)

$$X'^0 - \mathcal{E}\dot{X} = 0, \;\; X' - \mathcal{E}\dot{X}^0 = 0 \;\text{ at }\; \sigma = 0, \pi. \tag{13.16}$$

These equations have a rather surprising interpretation. To see this we express them in terms of light-cone coordinates on the world-sheet, that is $\sigma^{\pm} = \tau \pm \sigma$, which imply $\partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ and hence that $\partial_\tau = \partial_+ + \partial_-$ and $\partial_\sigma = \partial_+ - \partial_-$. Replacing the derivatives above by these $\partial_{\pm}$ derivatives the equations become

$$\begin{aligned} \partial_+ X^0 - \mathcal{E}\partial_+ X &= \partial_- X^0 + \mathcal{E}\partial_- X, \\ -\mathcal{E}\partial_+ X^0 + \partial_+ X &= \mathcal{E}\partial_- X^0 + \partial_- X. \end{aligned} \tag{13.17}$$

Solving these equations for $\partial_+ X^0$ and $\partial_+ X$ gives

$$\partial_+ \begin{pmatrix} X^0 \\ X \end{pmatrix} = \begin{pmatrix} \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} & \frac{2\mathcal{E}}{1-\mathcal{E}^2} \\ \frac{2\mathcal{E}}{1-\mathcal{E}^2} & \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ X \end{pmatrix}. \tag{13.18}$$

Splitting the X^0 and X into their left and right moving parts this equation becomes

$$\partial_+ \begin{pmatrix} X_L^0 \\ X_L \end{pmatrix} = \begin{pmatrix} \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} & \frac{2\mathcal{E}}{1-\mathcal{E}^2} \\ \frac{2\mathcal{E}}{1-\mathcal{E}^2} & \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} \end{pmatrix} \partial_- \begin{pmatrix} X_R^0 \\ X_R \end{pmatrix}. \tag{13.19}$$

Consider a situation where both X^0 and X have a N bc, i.e., $X'^0 = X' = 0$. Written in terms of $\partial_{\pm}$ derivatives the N b.c.s read

$$\partial_+ \begin{pmatrix} X^0 \\ X \end{pmatrix} = \partial_- \begin{pmatrix} X^0 \\ X \end{pmatrix}, \tag{13.20}$$

which after splitting into left and right parts reads

$$\partial_+ \begin{pmatrix} X_L^0 \\ X_L \end{pmatrix} = \partial_- \begin{pmatrix} X_R^0 \\ X_R \end{pmatrix}, \tag{13.21}$$

which in fact corresponds to the $\mathcal{E}$ equation above for $\mathcal{E} = \mathbf{1}$.

Let us now perform a T-duality transformation $X_L \to X_L$ and $X_R \to -X_R$ and use the definition $\tilde{X} = X_L - X_R$. Then the N bc equation for X^0 and X becomes

$$\partial_+ \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix}. \tag{13.22}$$

This equation confirms what we have seen above, namely that a T-duality flips a N bc to a D bc, or the other way around.

However, the purpose of the exercise done above is to argue that having a non-zero electric field given by $\mathcal{E}$ present when doing the T-duality will result in an equation which looks relativistic: This fact will then be used to claim that $\mathcal{E} = \beta$. The end result is hence that $\mathcal{E}^2 \leq 1$, a fact that will have profound implications for the Lagrangian and for how the electric field enters it (see next part of this lecture). So before turning to the construction of this Lagrangian let us finish the argument that tells us that $\mathcal{E} = \beta$.

Recall that doing a T-duality transformation in the standard situation with $\mathcal{E}^2 = 0$ results in a $D(p-1)$-brane sitting at rest at some point along the circle $S^1_{\tilde{R}}$ as we saw from the mode expansion of $\tilde{X}$ above. Turning on the electric field $\mathcal{E}$ then implies that the $D(p-1)$ starts to move around the circle with velocity $\beta = \mathcal{E}$. To see this, we write the last formula above in terms coordinates $(Y^0, \tilde{Y})$ that are a boosted version of $(X^0, \tilde{X})$, i.e.,

$$\partial_+ \begin{pmatrix} Y^0 \\ \tilde{Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_- \begin{pmatrix} Y^0 \\ \tilde{Y} \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} Y^0 \\ \tilde{Y} \end{pmatrix} = \begin{pmatrix} \gamma(X^0 - \beta\tilde{X}) \\ \gamma(-\beta X^0 + \tilde{X}) \end{pmatrix} = M(\beta, \gamma) \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix}. \tag{13.23}$$

This gives

$$\partial_+ \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix} = M^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M \partial_- \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix}. \tag{13.24}$$

If we now T-dualise back to (X^0, X) with (N, N) bc on the original circle S^1_R, this equation becomes

$$\partial_+ \begin{pmatrix} X^0 \\ X \end{pmatrix} = M^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ X \end{pmatrix}. \tag{13.25}$$

Doing the matrix multiplications give directly the following result

$$\partial_+ \begin{pmatrix} X^0_L \\ X_L \end{pmatrix} = \begin{pmatrix} \frac{1+\beta^2}{1-\beta^2} & \frac{2\beta}{1-\beta^2} \\ \frac{2\beta}{1-\beta^2} & \frac{1+\beta^2}{1-\beta^2} \end{pmatrix} \partial_- \begin{pmatrix} X^0_R \\ X_R \end{pmatrix}. \tag{13.26}$$

This is exactly the formula we derived above but then for $\mathcal{E}$ instead of the relativistic velocity β of the $D(p-1)$ brane moving on the $S^1_{\tilde{R}}$-circle. Thus we conclude that

$$\mathcal{E} = 2\pi\alpha' E = \beta. \tag{13.27}$$

In an analogues way we can establish the fact that using a non-zero magnetic field $F_{ij} = B_{ij}$

in the plane spanned by X^i and X^j instead of an electric one $F_{0i} = E_i$ the boost is replaced by a space rotation between the two directions involved.

13.2 Chapter 20: The non-linear Born-Infeld dynamics

There is a very strange, and not very often used, old result by Born and Infeld from 1934 that cures the infinite self-energy problem in electrodynamics. The Born-Infeld theory is a non-linear version of the ordinary linear Maxwell theory, and being non-linear it does not seem fundamental and cannot be part of a renormalisable QFT. However, in string theory the effective low-energy Lagrangian is certainly not renormalisable as a QFT theory, even if gravity is neglected (it is highly non-linear in any field appearing in the $\mathcal{L}^{eff}$). So one may in fact wonder if the somewhat strange Born-Infeld theory could play a role in string theory by being part of $\mathcal{L}^{eff}$. We will see below that this is indeed exactly what happens.

To get some understanding of the non-linear Born-Infeld theory we start by looking at electromagnetic field inside a material that have charges that can move and create effects like polarisation, screening etc. The standard way to treat this situation in EM is to start by introducing the usual vector potential A_μ and its field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Thus the Bianchi identities for the electric field $\mathbf{E}$ and the magnetic $\mathbf{B}$ are satisfied:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\partial_t \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0. \tag{13.28}$$

The equations of motion are sourced by the *free* currents and charges in the material that are not involved in the polarisation, screening etc. These latter effects are instead taken care of by introducing the "effective" fields $\mathbf{D}$ and $\mathbf{H}$ satisfying the ordinary-looking Maxwell equations

$$\nabla \cdot \mathbf{D} = \rho_{free}, \quad \nabla \times \mathbf{H} = \frac{1}{c}\mathbf{j}_{free} + \frac{1}{c}\partial_t \mathbf{D}. \tag{13.29}$$

Here the sources $(\rho_{free}, \mathbf{j}_{free})$ describe the charges and currents that move in the background of the polarised and screened medium. However, these dynamical equations are potentially non-linear due to the relation to the fundamental fields in the Bianchi identities

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B}), \quad \mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B}). \tag{13.30}$$

These relations are determined by experiments for the system under study.

In Lorentz covariant notation we can summarise these equations as

$$\partial_{[\mu} F_{\nu\rho]} = 0, \quad \partial_\nu G^{\mu\nu} = \frac{1}{c} j^\mu_{free}. \tag{13.31}$$

The field strengths $F^{\mu\nu}$ and $G^{\mu\nu}$ are related to $\mathbf{E}, \mathbf{B}$ and $\mathbf{D}, \mathbf{H}$, respectively, as usual in EM.

The Lagrangian description of these equations gives further insight. Since we don't know the non-linear relations also the Lagrangian is unknown. So a general variation of the action functional

$$S[A_\mu] = \int d^Dx \mathcal{L}(F_{\mu\nu}) + \frac{1}{c}\int d^Dx A_\mu j^\mu_{free}, \tag{13.32}$$

under δA_μ to find the field equations gives (dropping boundary terms)

$$\delta S[A_\mu] = \int d^Dx \frac{1}{2}\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\delta F_{\mu\nu} + \frac{1}{c}\int d^Dx j^\mu \delta A_\mu = \int d^Dx \Big(\partial_\nu (\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}) + \frac{1}{c} j^\mu_{free}\Big)\delta A_\mu \tag{13.33}$$

So the field equations obtained from $\delta S[A_\mu] = 0$ becomes identical to the ones above for $G_{\mu\nu}$ if we set

$$G^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}. \tag{13.34}$$

We should now try to understand how to construct general Lagrangians which are non-linear in the field strength $F_{\mu\nu}$. To respect Lorentz invariance $\mathcal{L}$ must be a function of the two scalar invariants

$$s := -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \tag{13.35}$$

$$p := -\frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma})F_{\mu\nu} = \mathbf{E}\cdot\mathbf{B}. \tag{13.36}$$

That these two invariants are the only ones is perhaps surprising if the argument is based on $F_{\mu\nu}$ but is clear using $\mathbf{E}$ and $\mathbf{B}$.

Born-Infeld theory

We are ready to construct the non-linear generalisation of Maxwell theory known as the Born-Infeld theory. Recall how we found that D-branes could be seen to move on the T-dualised circle if an electric field $\mathbf{E}$ was introduced on the D-brane before performing the T-dualisation (inversion of the circle radius). The velocity was found to be given by electric field by $\mathcal{E} = 2\pi\alpha' E = \beta$. The fact that $\beta < 1$ (special relativity) means therefore that also $\mathcal{E} < 1$ and that $\mathbf{E}$ has a maximal value.

This last property can be implemented in the Lagrangian in analogy with relativistic mechanics (the square root) as done by Born and Infeld

$$\mathcal{L}_{BI} = -\sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})} + 1. \tag{13.37}$$

First, this Lagrangian is clearly Lorentz invariant since a Lorentz transformation ${\Lambda_\mu}^\nu$ gives $\eta_{\mu\nu} + F_{\mu\nu} \to {\Lambda_\mu}^\rho {\Lambda_\nu}^\sigma(\eta_{\rho\sigma} + F_{\rho\sigma}) = {\Lambda_\mu}^\rho(\eta_{\rho\sigma} + F_{\rho\sigma}){\Lambda^{T\sigma}}_\nu$. The invarians of $-\det(\eta_{\mu\nu} + F_{\mu\nu})$ then follows from the product rule for determinants and that $\det\Lambda = 1$. But also $\det\Lambda = -1$ is OK since this determinant appears squared. An especially nice feature of this Born-Infeld Lagrangian compared to other generalisations is that $\mathcal{L}_{BI}$ can be used in any space-time dimension D.

A second property of $\mathcal{L}_{BI}$ is that since $\det(\eta_{\mu\nu} + F_{\mu\nu}) = \det(\eta^T_{\mu\nu} + F^T_{\mu\nu}) = \det(\eta_{\mu\nu} - F_{\mu\nu})$

the field strength must appear squared in one writes out the whole $\sqrt{-\det(\eta_{\mu\nu}+F_{\mu\nu})}$ as a power series in $F_{\mu\nu}$.

Thirdly, the form of $\mathcal{L}_{BI}$ has been designed to be precisely a non-linear generalisation of the ordinary Maxwell theory. This means that in a small-field expansion the first term must be the Maxwell term $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$. Clearly the addition of 1 to the square root means that the first term in the expansion is F^2. The way to perform the expansion is to write, in matrix form, $\det(\eta+F)=\det\eta\det(\mathbf{1}+F\eta^{-1})$ and perform the expansion in powers of $(F\eta^{-1})_\mu{}^\nu$.

In $D=4$ dimensions $\mathcal{L}_{BI}$ can be written as follows[45]

$$\mathcal{L}_{BI}^{D=4}=-\sqrt{1-2s-p^2}+1, \tag{13.38}$$

where s and p are defined above. It is then obvious that the first term is ordinary Maxwell theory since $\mathcal{L}_{BI}=s+\mathcal{O}(s^2,p^2)=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\mathcal{O}(F^4)$.

To see what the physical implications are we use the fact, mentioned above, that the Born-Infeld theory eliminates the infinity in the electron self-energy. This is easily demonstrated: (we denote energies in the two cases E_{EM} and E_{BI}, and $|\mathbf{E}|=E$ etc)

$$\text{Maxwell:}\ \ E_{EM}\sim\int_{r=0}^{\infty}d^3r\mathbf{E}^2\sim\int_{r=\epsilon}^{\infty}dr\,r^2(\frac{1}{r^2})^2\sim\frac{1}{\epsilon}\to\infty\ \ \text{as}\ \epsilon\to 0. \tag{13.39}$$

For the Born-Infeld theory this calculation changes dramatically since the energy is now given by $E_{BI}\sim ED$. This is also the case in Maxwell theory but there $\mathbf{D}=\mathbf{E}$. The reason the energy is $E_{BI}\sim ED$ comes from the fact that to make $\mathcal{L}_{BI}$ dimensionally correct we have to insert a constant, b with dimension $1/L^2$, as follows (for $\mathbf{B}=0$)

$$\mathcal{L}_{BI}^{D=4}=-b^2\sqrt{1-\frac{\mathbf{E}^2}{b^2}}+b^2. \tag{13.40}$$

In the self-energy context we are only interested in large field values in the $r\to 0$ limit. To study this we need the Born-Infeld Hamiltonian (recall from above that $\mathbf{D}$ is the "canonical momentum" of "velocity" $\mathbf{E}$)

$$\mathcal{H}_{BI}=\mathbf{E}\cdot\mathbf{D}-\mathcal{L}_{BI}=b^2\sqrt{1+\frac{D^2}{b^2}}-b^2. \tag{13.41}$$

Finally we can take the large field limit to get the Born-Infeld self-energy, the value of $\mathcal{H}_{BI}$,

$$\mathcal{H}_{BI}\sim b|D|\Rightarrow E_{BI}\sim b\int_{r=0}^{\infty}d^3r|D|\sim b\int_{r=\epsilon}dr\,r^2(\frac{1}{r^2})\sim\text{finite as}\ \epsilon\to 0. \tag{13.42}$$

Note that for a point charge inside a medium $\mathbf{D}(r)=-\nabla\frac{Q_{free}}{4\pi r}$. It is also important to find out what the physical meaning of b is. One can in fact compute $\mathbf{D}$ (the canonical

[45] In a very recent development a further one-parameter generalisation of the Born-Infeld theory was found that is also electro-magnetic duality invariant, see Bandos et al, hep-th/2007.09092.

momentum of $\mathbf{E}$) from the Lagrangian and solve the equation you get for $\mathbf{E}$. The result is

$$E^2 = D^2 \left(\frac{b^2}{b^2 + D^2} \right). \tag{13.43}$$

This equation is interesting since it shows that $|\mathbf{E}|$ cannot exceed the value b:

$$\text{Born-Infeld: } \ |\mathbf{E}| \leq b := E_{max}. \tag{13.44}$$

The value of $b = E_{max}$ is determined by the physical system under study.

The Born-Infeld self-energy can in fact the computed exactly (see BZ) for a point charge. Using $\int_0^\infty dx(\sqrt{1+x^4} - x^2) = \frac{(\Gamma(1/4))^2}{6\sqrt{\pi}}$ the **finite** result is (with an approximate value for the definite integral) it becomes in a string context (see below)

$$E_{BI}^{self-energy} \sim \frac{1}{4\pi} \cdot 1.748 \frac{Q^2}{\sqrt{\alpha' Q}}. \tag{13.45}$$

This should be compared to the usual **infinite** (as $a \to 0$) result in Maxwell theory

$$E_{Maxwell}^{self-energy} \sim \frac{1}{4\pi} \cdot \frac{3}{5} \frac{Q^2}{a} \text{ with a point like charge in the limit } a \to 0. \tag{13.46}$$

In string theory the Born-Infeld theory is relevant for Dp-branes as we saw above and must involve the only dimensionfull constant α' so

$$\mathcal{L}_{Dp} = -T_p(g_s)\sqrt{-\det(\eta_{mn} + 2\pi\alpha' F_{mn})}. \tag{13.47}$$

Several comments are needed:
1) The factor $2\pi\alpha'$ makes the whole expression under the square root dimensionless.
2) The term b^2, or $+1$, in previous versions of the Lagrangian is not present here since setting $F_{\mu\nu} = 0$ should give the energy stored in the D-brane without any fields on it.
3) The stored energy in 2) comes from the tension as indicated by the coefficient $T_p(g_s)$ with dimension $[energy/volume] = L^{-(p+1)}$.
4) In the context of the string Polyakov path integral the comment was made that the dependence of g_s is determined by the Euler number $\chi = 2 - 2g - b$ ($g = genus = \#_{loops}$ and $b = \#_{boundaries}$) as $(g_s)^{-\chi}$. Thus computing the coefficient of the Einstein-Hilbert action in spacetime at tree level, i.e., from the genus zero closed Riemann surface having $g = 0$ and thus $\chi = 2$, we get the result $\frac{1}{g_s^2} \int d^D x R$. D-branes are in a similar way related to open strings and hence Riemann surfaces with one boundary. At tree level (g=0) this gives $\chi = 1$ and therefore

$$T_p(g_s) \sim g_s^{-1}. \tag{13.48}$$

Using α' to get the dimension correct, the final answer is

$$T_p(g_s) = \frac{1}{g_s (2\pi)^p (\alpha')^{(p+1)/2}}. \tag{13.49}$$

Comments:
1) The $g_s = g_{YM}^2$ dependence of the brane tension $T_p(g_s)$ is the one obtained for **instantons** or **solitons** in ordinary Yang-Mills field theories. These objects are solutions of the classical field equations which are considered as non-perturbative from a QFT point of view (similar to bound states in QFT, see Peskin and Schroeder pages 148 - 153). The characteristic feature of such solutions is that their energy goes to infinity as the coupling constant goes to zero.
2) The results obtained are all for Maxwell like fields, i.e., $F_{\mu\nu}$ is not a matrix (as in Yang-Mills theory) which makes the square root rather ease to handle. In the case of an N-stack of D-branes we know from previous discussions that the Maxwell theory is replaced by a Yang-Mills theory based on the gauge group $U(1) \times SU(N)$. Unfortunately, this makes the development of a Born-Infeld like theory very much more complicated and it is in fact not yet known how to obtain such a theory in a constructive way.
3) It is interesting to note that the first of the previous comments does not seem to have an analogue in M-theory. The second one, on the other hand, is believed to have an analogue but it is not understood exactly how it would work mathematically. There are results for $M2$-branes ending on one $M5$-brane leading to a closed string theory (instead of a Maxwell type field theory) living on the $M5$-brane. In the case of a stack of $M5$-branes this **little string theory** is based on gauge groups which are simply-laced, i.e., A_n, D_n and E_8, known as the ADE classification. Since the "little string theory" consists of closed strings it should contain gravitons which is not possible on an M-brane. This, and many other strange features, has made it very hard to understand and construct explicit versions of such little string theories. The final answer is believed to involve a supersymmetric CFT theory in the 1+5 dimensional world-volume of the $M5$-brane denoted $\mathbf{(2,0)CFT_6}$[46]. A funny aspect of such theories is that they must contain a gauge field that couples to a string, that is a Kalb-Ramond field $B_{\mu\nu}$. Furthermore, this field must have a self-dual field strength $H^+_{\mu\nu\rho}$ and hence there is no simple Lagrangian formulation for it in 1+5 spacetime dimensions (similar to the situation for $G^+_{\mu_1\ldots\mu_5}$ in type IIB supergravity in 1+9 dimensions). Another huge problem is that it is enormously tricky to write down interaction terms for any kind of Kalb-Ramond fields.
4) Dp-branes can also have a Polyakov type formulation which can be generalised to curved Dp-branes described by the **Dirac-Born-Infeld** action[47]

$$S^{(p)}_{DBI} = -T_p(g_s)\sqrt{-\det(g + 2\pi\alpha'\mathcal{F})}, \tag{13.50}$$

which combines the Nambu-Goto and Born-Infeld ideas. So, here g refers to the induced metric $g_{mn} = \partial_m X^\mu \partial_n X^\nu \, g_{\mu\nu}$ and $\mathcal{F}$ to the stringy field strength $\mathcal{F}_{mn} = F_{mn} - B_{mn}$. The Polyakov version of these $S^{(p)}_{DBI}$ actions are in general quite tricky to derive (see last footnote).

[46] It is known from a theorem proved by W. Nahm in Nucl. Phys. B135 (1978) p. 149 (see also p. 6 in Cordova et al hep-th/1602.01217 or S. Minwalla hep-th/9712074) that there are no superconformal theories in higher dimensions than 6.
[47] See, e.g., Cederwall et al hep-th/9606173.

14 Lecture 16

The purpose of this final lecture is to briefly introduce some ideas that are useful in bringing string theory closer to real physical systems: **brane world scenarios** and the **AdS/CFT correspondence**. Both require a lot deeper understanding of string theories than is provided in this course so the following introductions to these subjects will be brief and many results will not be derived in detail. It should be said in this context that the AdS/CFT correspondence is far from being proved but an enormous amount of scattered results indicate that it is true. It was discovered by Juan Maldacena in 1997, see hep-th/9711200, a paper which now (Dec. 2020) has over 20.000 citations. There have been, however, holographic ideas floating around in physics prior to Maldacena's paper but his paper was certainly the first step towards a new and very deep aspect of string theory.

The swampland program

A more recent attempt to connect quantum gravity, or string theory, to the Standard Model and other kinds of theories used in our description of the universe is the *swampland program*. In this approach to extract physics from quantum gravity one aims at formulating conditions on field theories without gravity which, if satisfied, make it possible to couple them to quantum gravity in a consistent way. Some of these conjectures can be (almost) proven from string theory but most cannot so these conditions are generally referred to as **swampland conjectures**. If a field theory satisfies all these conjectured conditions it ends up in the "nice area", i.e., the **landscape**, otherwise it belongs to the huge number of bad theories in the swampland. This philosophy is also highly relevant for the applications of AdS/CFT to strongly coupled condensed matter systems. We end this brief account with some examples of swampland conjectures:

Example 1: The weak gravity conjecture (WGC): Gravity is the weakest of all forces[48].
Example 2: There are no global symmetries. This is actually a known feature of string theory and is, e.g., relevant when discussing the $B - L$ global symmetry of the Standard Model.
Example 3: There are no stable non-supersymmetric AdS vacuum solutions[49]. This is presently under heavy debate but no accepted counter examples are yet known.
Example 4: There are no stable (or semi-stable) de Sitter solutions at all[50]. This conjecture will clearly have profound implications if true! Remember that our universe is known from observations to be de Sitter.

[48] See "The string landscape, black holes and gravity as the weakest force" by Arkani-Hamed, Motl, Nicolis and Vafa, hep-th/0601001.

[49] See "Non-supersymmetric AdS and the swampland" by Ooguri and Vafa, hep-th/1610.01533.

[50] See "de Sitter space and the swampland" by Obied, Ooguri, Spodyneiko and Vafa, hep-th/1806.08362.

14.1 Chapter 21: String theory and particle physics (briefly)

Here the idea is to present a string theory setup that generates several features possessed by the Standard Model used in elementary particle physics in 1+3 dimensional Minkowski space. To do this we will consider string theories with the following properties[51]:

1) Superstrings are needed to get spacetime fermions. Note that there are no Yang-Mills fields in the theory in $D = 10$ unless we use heterotic strings.

2) We need six compact dimensions to get from $D = 10$ to $D = 4$. To make it as simple as possible we use T^6 and define the split $x^M = (x^\mu, x^m)$ with

$$x^m = (x^4, ..., x^9) \in (S^1)^6 = T^6 \text{ where all circles have the same radius R: } x^m \sim x^m + 2\pi R. \tag{14.1}$$

3) We need several different N-stacks of D-branes to get at least a $U(1) \times SU(2) \times SU(3)$ gauge group. Note that each non-abelian factor comes with a separate $U(1)$ factor.

4) Since we would like to use the Yang-Mills theories living on the D-brane stacks to give the usual Standard Model gauge fields, our 1+3 dimensional spacetime must fit into the Dp-branes, i.e., $p \geq 3$. We will use $D6$-branes, and hence type IIA strings, for reasons explained below. Note that there are therefore no Yang-Mills fields in this theory in $D = 10$.

5) These different stacks of $D6$-branes must have the 1+3 dimensional spacetime as common directions, i.e., they must be intersecting D-branes with 1+3 spacetime intersections. As an example consider the following two stacks (using the 10=4+6 split $M = (\mu, m)$):

$M =$	+	-	2	3	4	5	6	7	8	9
$D6_1$	N	N	N	N	N	D	N	D	N	D
$D6_2$	N	N	N	N	D	N	D	N	D	N

Since N bc represent a direction within a Dp-brane we see that X^M for $M = \mu = +, -, 2, 3$ are common to both of these stacks of $D6$-branes which, furthermore, are orthogonal in the remaining 6 torus directions. Note the alternating assignments of N and D b.c.s on T^6. We will also place the D6-branes at the "origin" of T^6, i.e., the $D6_1$-branes sit at $x^5 = x^7 = x^9 = 0$ and $D6_2$-branes sit at $x^4 = x^6 = x^8 = 0$.

6) Three (or more) families may be obtained using strings that wind around the target space circles in T^6.

[51]The duality relations between different string theories, and M-theory, indicate that Standard Model like models should be possible to find in all string/M theories. And indeed, several other type of derivations of Standard Model physics are known. Among the more intriguing ones is the compactification of $D = 11$ M-theory on so called $G2$-manifolds leading to supergravity theories in $D = 4$ AdS or Minkowski space with **one** supersymmetry which is what is needed for the $MSSM$ beyond the Standard Model theory.

This particular setup makes it possible to set $T^6 = T^2(45) \times T^2(67) \times T^2(89)$, where the compact dimensions of each two-torus are given in the bracket. We can then represent each two-torus by a square with its sides identified and the axes corresponding to **one** dimension from each of the two stacks of D-branes, the x-direction from the $D6_1$-branes and the y-direction from the $D6_2$-branes. So an open string stretching from $D6_1$ to $D6_2$ will appear in each of these three T^2 diagrams as a curve from some point on the x-axis to the some point on the y-axis (avoiding the origin to make the situations clear). As has been discussed previously the mode expansion of such DN or ND-strings are tied to the intersection point and will in fact stay close to it due to their own tension. The low energy fields will therefore "live on the intersection" which is our 1+3 dimensional spacetime.

Also, since the orthogonal stacks of D6-branes have different gauge groups the ends of these strings will have charges corresponding to different representations of the gauge group in question. If we denote as [12] a string that starts at the $D6_1$, which we choose now to be a 2-stack, and ends at the $D6_2$, which we decide is a 3-stack, then open superstring massless Ramond states will be spacetime spin 1/2 fields charged as indicated by the indices on ψ^a_A where a is a 2-dimensional irrep of $SU(2)$ and A is a 3-dimensional irrep of $SU(3)$. This set of irreps is possessed by the left-handed $(u, d)_L$ type quarks in each family of the Standard Model.

These charges are actually coming from the collection of open strings with one end on each of the branes in the **weak** 2-stack and the other end on each of the branes in the **strong** 3-stack in all possible ways. That is, if we denote the two branes in the 2-stack as $+$ and $-$, and the three branes in the 3-stack as **blue**, **green** and **red** we get all the six possible fermionic fields in ψ^a_A. Furthermore, since the two ends of a string have opposite charges we define a $\sigma = \pi$ end to have charges **red**, **blue** and **green** and a $\sigma = 0$ end to have charges **anti-blue**, **anti-green** and **anti-red**. The former set of three charges transform as the irrep $\mathbf{3}$ under the strong, or color, group $SU(3)$ and latter set as the irrep $\bar{\mathbf{3}}$. The situation is the same for the 2-stack where the charges $(+, -)$ transform as the irrep $\mathbf{2}$ under the weak $SU(2)$. In QM we know this irrep as spin-1/2. One difference between $SU(2)$ and $SU(3)$, however, is that the anti-$(+, -)$ irrep is the same as $(+, -)$.

This way of assigning charges to fermion fields in spacetime can be carried over to the NS sector and its massless vector fields. The counting then leads to N^2 gauge fields and thus to the gauge groups we have identified before, namely $U(1) \times SU(N)$. We can then immediately identify one interesting fact: Since we must use one 3-stack and one 2-stack it seems unavoidable to get **two** $U(1)$ gauge fields. This might mean one of two things:
1) Either there exist more complicated versions of these string models that remove the extra Maxwell field (which it does) with the remaining one playing the role of **hypercharge** gauge field, or
2) the Standard Model really must be extended with another Maxwell field to be consistent with string theory and hence with quantum gravity (such $U(1)$ fields are being looked for

e.g. at CERN and analysed theoretically in many many papers).

Families:
In order to get also the remaining fermions in the Standard Model (leptons etc) one has to introduce a number of other D6-branes in a rather complicated way. This can however be done. The issue which is of a slightly different nature is how to get more than one family, and perhaps also to understand why the number of families is exactly three. The number of families can be connected to winding numbers on the three 2-torii in T^6 used above.

Higgs:
Spontaneous symmetry breaking is easy get by separating the branes in some of the stacks (as discussed previously) but getting the potential needed for the Higgs field is tricky.

Some different approaches giving physics in $D = 4$ Minkowski space:
1) Type IIA on T^6 with $D6$ -branes $\rightarrow$ SM without susy.

2) Het($E_8 \times E_8$) on Calabi-Yau $\rightarrow$ MSSM (one susy)

3) M-theory on G2-manifolds with singularities $\rightarrow$ MSSM (one susy)

Moduli stabilisation:
This is one of the most important issues in string theory when trying to make contact with ordinary physics. It is also one of the currently most studied questions as part of the search for **de Sitter** solutions[52] and more generally as part of the **swampland program**.

We will here discuss the moduli stabilisation problem in one of the most simple, but yet important, situations: How is the radius of a compact circle dimension determined by the string? A follow-up question is: Can this be generalised to **all** other moduli?

Consider the Kaluza-Klein compactification we studied in the first week of the course:

$D = 5$ gravity on $S^1_R \rightarrow D = 4$.

The radius is actually a scalar field $R(x)$ in $D = 4$ since the metric ansatz for the compactification reads (using the 5=4+1 split $x^M = (x^\mu, y)$ and setting $g_{\mu 5} = 0$)

$$ds^2_{D=5} = g_{MN} dx^M dx^N = g_{\mu\nu} dx^{\mu\nu} + g_{55}(x) dy^2. \tag{14.2}$$

In the last term we can let $y \in [0, 2\pi R_0]$ with a fixed R_0. Then the physical radius of the circle is (remember that g_{55} is dimensionless)

$$R(x) = 2\pi R_0 \cdot g_{55}(x). \tag{14.3}$$

[52] A very interesting comment on this problem appeared very recently in Gia Dvali hep-th/2012.02133: There may be a fundamental clash between string theory, being defined in terms of its perturbation expansion and S-matrix, and de Sitter, which does not have an S-matrix since its space is compact.

So the question is how to get a fixed value of the radius, that is $\langle R(x)\rangle$. This situation is very similar to how the Higgs effect works so we need to find a potential $V(R(x))$ for the $D=4$ scalar field $R(x)$. However, this compactification does not give rise to a potential (see home problem 1). In this case all values of $\langle R(x)\rangle$ are equally good and thus provides a parameter in the space of solutions, that is, a **modulus**. Can we improve on this situation and get a potential?[53]

To get a non-zero potential $V(R(x))$ we go up one dimension and consider KK from $D=6$ on Σ_2 to $D=4$ where the 2-dimensional compact internal space Σ_2 is a Riemann surface with genus g (a multi-hole torus) and Euler number $\chi = 2-2g$. Without going through the rather long calculation starting from (with the 6=4+2 split $X^M = (x^\mu, y^m)$ and $g_{\mu n} = 0$)

$$ds^2_{D=6} = g_{MN}dx^M dx^N = g_{\mu\nu}(x)dx^\mu dx^\nu + R(x)\bar{g}_{mn}(y)dy^m dy^n, \tag{14.4}$$

where $\bar{g}_{mn}$ is a fixed sized metric on Σ_2, we just state the result that we now do get a potential:

$$V_g(R) = -a_g\frac{\chi}{R^4}, \quad a_g > 0. \tag{14.5}$$

So using the theory of Riemann surfaces (the first implication below uses $\chi = 2-2g$) we get information about the potential (second implication) and the behaviour of R:

$$\begin{aligned}
g = 0 \ &\Rightarrow \chi = 2 \Rightarrow V_g(R) < 0 \Rightarrow R \to 0,\\
g = 1 \ &\Rightarrow \chi = 0 \Rightarrow V_g(R) = 0 \Rightarrow R = \text{modulus},\\
g > 1 \ &\Rightarrow \chi < 0 \Rightarrow V_g(R) > 0 \Rightarrow R \to \infty.
\end{aligned} \tag{14.6}$$

Although the situation improved since we did get a non-zero potential it did not stabilise the radius R, i.e., this $V(R(x))$ does not have a global or local minimum giving a finite value for $\langle R(x)\rangle$. To obtain a potential with this property it must contain more terms. One possibility is to use **fluxes**. (This is a very hot research topic today in exactly this context.) The word "flux" refers here to a gauge field, that is a p-form, which when integrated over a closed p-surface gives an integer, the flux through the surface. Using Gauss' law in $d=3$ to compute the magnetic charge inside a closed 2-surface is a well-known example: $Q_m = \frac{1}{4\pi}\int_{S^2}\frac{1}{2}F^{ij}da^{ij}$. Let us denote, as usual, the integrand as B_2. Then

$$\frac{1}{4\pi}\int_{\Sigma_2} B = n, \quad |B| \sim \frac{n}{R^2}. \tag{14.7}$$

This flux $|B|$ gives a contribution to the energy, in fact the potential energy, $E \sim R^2B^2$ where B^2 is the usual energy density in EM and R^2 comes from the "volume" of Σ_2. Thus the new potential is

$$V_{flux}(R) = a_f\frac{n^2}{R^6}, \quad (a_f > 0). \tag{14.8}$$

[53] It has become a tradition in string theory to call also scalar fields VEVs that have a non-zero potential for mudulii.

The final R^6 in the denominator arises from $R^2 B^2 \sim \frac{n^2}{R^2}$ multiplied by $\frac{1}{R^4}$ coming from the fact that one has to make a Weyl-transformation ($g_{\mu\nu} \to f(R) g_{\mu\nu}$) to end up with an Einstein-Hilbert term in $D = 4$ without a function of R multiplying it.

Thus the new potential is

$$V_{new}(R) = -a_g \frac{\chi}{R^4} + a_f \frac{n^2}{R^6}, \;\; (a_g, a_f > 0). \tag{14.9}$$

The situation (when $n \neq 0$) for stabilisation has changed:

$$\begin{aligned}
&g = 0 \;\; \Rightarrow \chi = 2 \Rightarrow V_{new}(R) \;\Rightarrow \text{one global minimum for finite R} \Rightarrow \;\text{R stabilased!},\\
&g = 1 \;\; \Rightarrow \chi = 0 \Rightarrow V_{new}(R) > 0 \Rightarrow \text{global minimum as } R \to \infty,\\
&g > 1 \;\; \Rightarrow \chi < 0 \Rightarrow V_{new}(R) > 0 \Rightarrow \text{global minimum as } R \to \infty.
\end{aligned} \tag{14.10}$$

de Sitter universe? The current attempts to obtain a de Sitter universe in string theory rely heavily on proving that all modulii can be fixed at values where the potential gives a negative cosmological constant. The swampland program seems to have a lot say here but, unfortunately, mostly in a negative way.

14.2 Chapter 23: The AdS/CFT correspondence (briefly)

In his 1997 paper *The large-N limit of superconformal field theories and supergravity"* Juan Maldacena gave a proposal for the answer to the question:

Question: There seems to exist two different descriptions of D-brane physics without gravity. What does this mean?

Let us review these two descriptions giving *non-gravitational* physics in the case of $D3$-branes[54]:

1) **Open string picture: open strings ending on N-stacks of D3-branes**
a) $U(N) = SU(N) \times U(1)$ Yang-Mills theory.
b) Conformal invariance in 1+3 dimensions, i.e., $SO(2,4)$ (or superconformal).
c) Dimensionless parameters N and g_{YM}.
d) Relevant objects: Gauge invariant operators like $\mathcal{O}_{F^2} = Tr(F_{\mu\nu}F^{\mu\nu})$.

2) **Closed string picture: Solutions in supergravity**
The basic parameters in superstring theory are: $l_s = \sqrt{\alpha'}$ and g_s together with size parameter R when we consider compactifiactions as we do here, namely AdS solutions. Note that the parameter R determines the size of both the spacetime AdS and the internal manifold which is a sphere in the discussion here. This kind of supergravity solutions have not been discussed before in this course. What we need to know here is that in, e.g., by setting $F_{\mu\nu\rho\sigma} \sim \epsilon_{\mu\nu\rho\sigma}$ in $D = 11$ supergravity one rather easily obtains a $D = 4$ a maximally symmetric solution which is $AdS_4 \times S^7$. In type IIB a similar procedure leads to an $AdS_5 \times S^5$ solution. In all such cases (there are several others) the size of both factor manifolds is given by the same parameter R as noted above. This is sometimes a problematic property known as the **scale separation problem**. Of course, we also assume that the Newton's constant is given in terms of these string parameters as shown previously in the course. The relevance of these comments become clear when we now analyze the solution of interest here.

Let us return to the charged brane solutions in supergravity discussed previously in the course. Remember that supergravity comes from the (NS, NS) sector of the closed superstring. This time the discussion concerns the 1+3 dimensional solution of type IIB supergravity: (using the split $10 = 4 + 6$ with $x^M = (x^\mu, y^m)$, and $r = |y|$)

$$ds^2(p = 3) = H^{-1/2}(r)dx^2 + H^{1/2}(r)dy^2, \quad H(r) = 1 + \frac{r_0^4}{r^4}, \tag{14.11}$$

$$G_5 = \frac{1}{g_s} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \, dH^{-1}. \tag{14.12}$$

where both dx^2 and dy^2 are flat (Lorentzian and Euclidean, respectively).

[54]The fact that D-branes correspond to solutions in supergravity with Ramond-Ramond charges was discovered in 1995 by Polchinski, see hep-th/9510017.

In order to check the properties of the solution

$$ds^2(p=3) = H^{-1/2}(r)(-dt^2 + d\mathbf{x}^2) + H^{1/2}(r)(dr^2 + r^2 d\Omega_5^2), \quad H(r) = 1 + \frac{r_0^4}{r^4}, \qquad (14.13)$$

close to the horizon we take $r \to 0$: $(H(r) \to \frac{r_0^4}{r^4})$

$$ds^2(p=3) \to ds^2|_{horizon} = (\frac{r_0}{r})^{-2}(-dt^2 + d\mathbf{x}^2) + (\frac{r_0}{r})^2(dr^2 + r^2 d\Omega_5^2). \qquad (14.14)$$

Now we change variables to $z = r_0^2/r$. This gives (note that $dr^2/r^2 = dz^2/z^2$)

$$ds^2|_{horizon} = \frac{r_0^2}{z^2}(-dt^2 + d\mathbf{x}^2) + r_0^2 \frac{dz^2}{z^2} + r_0^2 d\Omega_5^2. \qquad (14.15)$$

If this metric is written $ds^2|_{horizon} = \frac{r_0^2}{z^2}(-dt^2 + d\mathbf{x}^2 + dz^2) + r_0^2 d\Omega_5^2$ we see that the first part is the metric of AdS_5 and the second one the metric of S^5. Thus

$$ds^2|_{horizon} \sim ds^2(AdS_5 \times S^5). \qquad (14.16)$$

Comment: The AdS_5 metric that arises close to the horizon above is a particularly simple version, known as the **Poincaré metric**. It does, however, not cover the whole of the manifold AdS_5. The simplicity becomes clear when computing the affine connection and the Riemann tensor. This metric has its boundary at $z \to 0$ (the horizon above was obtained for $z \to \infty$) which shows that the AdS boundary geometry is $D = 4$ Minkowski space.

Exercise: Compute the Riemann tensor of the Poincaré metric above using $x^\mu = (x^0, x^1, x^2, x^3, z)$.

Comment: The appearance of $AdS_5 \times S^5$ at the horizon demonstrates the close relation of this solution to **solitons**[55]. A simple example of a soliton is the solution of the Klein-Gordon field equations in 1+1 dimensions with a potential that has two local minima at $\phi_\pm = \pm\phi_0$: Solitons are then solutions which interpolate between these two minima: $\phi(x \to +\infty) \to \phi_+$ and $\phi(x \to -\infty) \to \phi_-$. Such solutions have energy stored in the region around the origin where the function $\partial_x \phi(x)$ is non-zero. The supergravity solution discussed here interpolates in a similar fashion between the two background solutions $D = 10$ Minkowski for $r \to \infty$ and $AdS_5 \times S^5$ for $r \to 0$. The stored energy in the D-brane supergravity solution is given by the mass M (see below).

The AdS/CFT correspondence:

The task now is to understand how features on one side enter the other side of the correspondence. Let's start with the conformal symmetry $SO(2, 4)$.

[55]Solitons were discovered by J.S. Russell in Edinburgh in 1834 when riding along a narrow channal he saw a solitary wave top moving for a long distance seemingly without loosing energy.

A) $SO(2,4)$: That this symmetry is present on both sides is clear from what has been said above. For the $D3$-brane it is the conformal symmetry and close to the horizon it is the isometry of AdS_5 . Note also the curious fact that the boundary of $AdS5$ is the *compactified* version of 1+3 dimensional Minkowski space. This word *compactified* is not related to Kaluza-Klein but means instead that all points at spatial infinity in $D = 4$ Minkowski space are identified thereby changing its topology to $\mathbf{R} \times S^3$, while remaining flat. This can be checked by taking $z \to 0$ above.

B) From before we know that $g_s = g_{YM}^2$.

C) We must also relate R in the supergravity solution $AdS_5 \times S^5$ to the parameters on the other side N and g_{YM}. Since these latter ones are both dimensionless it is actually the dimensionless ratio R/l_s we should consider. Here arises a crucial question: Are we using

$$R \gg l_s \Rightarrow \text{strings moving in a large } AdS_5 \times S^5, \tag{14.17}$$

or

$$R \ll l_s \Rightarrow ?????(\text{hard to describe}) \tag{14.18}$$

Clearly it is the first case we are dealing with in the current discussion of AdS/CFT. However, the other case is extremely important too as we will see below. So, how do we find the relation between R/l_s and N and g_{YM}?

We know that in $D = 10$ Newton's potential from the brane solution above is given by $V(r) = -\frac{G_N^{(10)}M}{r^4}$ which is dimensionless since it is related to the g_{00} component of the metric. Define then r_0 by

$$r_0^4 = G_N^{(10)}M. \tag{14.19}$$

In terms of stringy parameters we know from before that

1) $G_N^{(10)} = g_s^2 l_s^8$, and

2) for an N-stack of $D3$-branes (assuming this is the source of the solution) that the mass is $M = N \cdot T_3$ where the $D3$-brane tension is $T_3 = 1/(g_s l_s^4)$. Thus

$$r_0^4 = g_s N l_s^4 \Rightarrow \frac{r_0^4}{l_s^4} = g_s N = g_{YM}^2 N := \lambda. \tag{14.20}$$

Here we need two clarifications: 1) r_0 is the parameters giving the scale, or "size", of the geometry of the supergravity solution, and thus of both factors in the close horizon geometry $AdS_5 \times S^5$. Hence $r_0 := R$, the parameter used above. Thus

$$(\frac{R}{l_s})^4 = g_{YM}^2 N := \lambda. \tag{14.21}$$

We then see that $R \gg l_s$ implies $g_{YM}^2 N := \lambda \gg 1$. The second clarification therefore concerns λ and the meaning of it.

The 't Hooft parameter λ:

That $g_{YM}^2 N := \lambda$ naturally appears in the above discussion is very interesting. λ is called the 't Hooft parameter and was introduced by 't Hooft when trying to develop an alternative to the usual perturbation expansion in terms powers of g_{YM}, in particular for $g_{YM} \gg 1$ when this expansion breaks down. The idea is as follows. Consider an $SU(N)$ Yang-Mills theory with a field strength $F = dA + g_{YM} A^2$ (schematically). By absorbing one factor of g_{YM} into the gauge field A_μ the Lagrangian density becomes (the trace is over the Lie algebra matrices)

$$\mathcal{L} = -\frac{1}{4g_{YM}^2} Tr\, F_{\mu\nu}F^{\mu\nu} = -\frac{N}{\lambda} Tr\, F_{\mu\nu}F^{\mu\nu}, \quad \text{where now } F = dA + A^2. \tag{14.22}$$

From the last form of the Lagrangian we conclude that the propagator is $\sim \frac{\lambda}{N}$ while the three-point and the four-point vertices are both proportional to $\frac{N}{\lambda}$. Any Feynman diagram will then behave as

$$\text{Diagram} \sim (\frac{\lambda}{N})^E \cdot (\frac{N}{\lambda})^V \cdot N^F, \tag{14.23}$$

where E (edges) is the number of propagators, V the number of vertices and F is the number of $faces$. This last concept corresponds to a loop inside a diagram and that it gives a factor N is seen as follows: Recall that since the gauge field A_μ is Lie algebra valued we can express it in some irrep (the lower i index, and anti-irrep the upper j index) of $SU(N)$ as

$$A_\mu = A_\mu^a\, (T^a)_i{}^j \Rightarrow \text{Draw the propagator as two // lines with arrows in opposite directions.} \tag{14.24}$$

Any loop in a diagram will then appear as a closed line with an arrow on it. Such a closed index-line generates a trace in the expression for the loop hence the factor N^F for the general diagram above with F loops.

What 't Hooft realized next was that this result can be expressed as a Riemann surface genus expansion which we already have encountered in the context of the string loop expansion. In the present case we have

$$(\frac{\lambda}{N})^E \cdot (\frac{N}{\lambda})^V \cdot N^F = N^{V-E+F}\, \lambda^{E-V} = N^\chi\, \lambda^{E-V}, \tag{14.25}$$

where we have made use of the observation that $\chi = V - E + F$ is just the Euler number for a Riemann surface (a multi-genus 2-torus). It is then possible to define a perturbative genus expansion in $1/N$ for large N and fixed λ of any size. In terms of g_{YM} this new series may even contain non-perturbative information.

We can now summarise the relations between the $AdS_5 \times S^5$ type IIB string parameters $l_s = \sqrt{\alpha'}, g_s, R$ and the Yang-Mills ones g_{YM}, N found above:

$$g_s = g_{YM}^2, \quad (\frac{R}{l_s})^4 = g_{YM}^2\, N = \lambda. \tag{14.26}$$

Comment: Using $G_N^{(10)} = g_s^2 l_s^8$, derived in string theory above, and the definition of the Planck length $G_N^{(10)} = l_P^8$ the above relations gives rise to another interesting relation (see below):

$$(\frac{l_P}{R})^4 = \frac{1}{N}. \tag{14.27}$$

Having established the basic relations between the parameters on the two sides we can start asking what they mean and how they can be used in actual physics applications. The discussion above concerns the identification of two "classical" objects, the D-branes in open string theory and the extremal charged brane solutions in supergravity from closed strings. So far there is nothing new compared to the work of Polchinski who discovered this relation. What Maldacena added was that it must be possible to also identify the physics of the low energy excitations in the two cases.

The crucial parameter in the following is the 't Hooft parameter $\lambda = g_s N = g_{YM}^2 N$. We will discuss the physics of the *low energy* excitations in the two limits $\lambda = g_s N \ll 1$ and $\lambda = g_s N \gg 1$ and try to find the physics that is disconnected from gravity (that is, the Yang-Mills sector). One key point here is that low energy now refers to energies much smaller than the Planck energy while in the previous discussion the energy did not play a role at all. Energy is normally related to scattering amplitudes and particle momenta and not to solutions of field equations. Also important here is "taking the limit in energy" must be done in terms of a dimensionless parameter, namely $E\, l_s$ in the D-brane case and $E\, l_p$ in the supergravity case.

$\lambda = g_s N \ll 1$: The excitations here are clearly the $U(N)$ gauge fields of stack of D-branes. This field theory is weakly coupled when $\lambda = g_s N = g_{YM}^2 N \ll 1$ for any value of (non-zero) N. There is a gravitational field generated by the stack which can have its own excitations. These excitations interact with themselves and with the matter fields (here the Yang-Mills fields). The strength of these interactions is determined by $G_N^{(10)}$, or l_p, so we must design another limit that takes these interactions to zero while keeping the Yang-Mills interactions small but finite. The gravitational interactions are however given at low energies by E/E_p which is very small. This is also true for the interactions between gravity and the fields on the branes. Thus the physical system in this case is just a weakly interacting Yang-Mills theory on the branes plus free gravity excitations in the bulk.

$\lambda = g_s N \gg 1$: The gravitational potential between the branes in the stack is governed by R which we determined above to be $(R/l_s)^4 = g_s N = \lambda \gg 1$. This strong gravity will collapse the stack of D-branes into a bound state which is naturally described as a black brane solution of the field equations, here the ones in supergravity coming from the closed strings. The low energy excitations in this case can only be closed string field excitations on the background of the black brane-solution.

There are two fundamentally different situations that one must analyse carefully:

1) $R \gg l_s$ and 2) $R \ll l_s$, or $\lambda \gg 1$ and $\lambda \ll 1$. However, both systems above should exist for any value of λ, or of g_s and N, which means that there is a duality, or correspondence, between them for any values of these parameters. The problem is that this statement is very hard to check since one system is simple in one of the λ limits and the system is simple in the pother limit. We will now make this more precise and useful by defining a particular kind of dubble limit: Large N and low energy.

1) $R \gg l_s$: This is the case studied by Maldacena in the limit of large N. Here the Yang-Mills system is strongly coupled (λ fixed but large) but the other side can be made weakly coupled and therefore useful computationally. The above relations tell us immediately that when $N \to \infty$ the string coupling constant $g_s \to 0$. This eliminates the g_s power series leaving only the first term. The other limit of low energy then implies that $\alpha' E^2 \to 0$. The implications of these two limits is best seen by looking at the general structure of the low energy effective action

$$S_{eff} = \Sigma_{m,n} (\alpha')^m (g_s)^n \, S_{eff}^{m,n}, \tag{14.28}$$

where the g_s power series corresponds to the string loop expansion and the α' one to the world-sheet loop expansion which corresponds in turn to an expansion of S_{eff} in powers of the Riemann tensor and other fields with derivatives in them. In fact, the actual dimensionless expansion parameter in this is $\alpha' E^2$ or $\alpha' \partial_\mu \partial_\nu$. So we conclude that in S_{eff} only the first term gives a significant contribution and all loop and higher derivative terms can be neglected. The remaining theory has weakly coupled Einstein-type classical gravity (for large but finite λ).

2) $R \ll l_s$: In this limit the situation for the two systems is the opposite one. The Yang-Mills system since λ is small but the other gravity side is strongly coupled which is seen by the fact that $R \ll l_s$.

The Maldacena correspondence described above is somewhat unclear about how the correspondence can be used since it is not specified in a mathematical way. This necessary additional information is provided stringent form of the correspondence developed by Witten[56], Gubser, Klebanov and Polyakov[57]. It is stated in terms of the field theory partition functions (path integrals) on the entire AdS, i.e., including the boundary. Then the correspondence can be expressed as the equality

$$Z_{AdS}(\bar{\phi} = J) = Z_{CFT}(J). \tag{14.29}$$

Here the RHS is the generating functional of correlation functions of gauge invariant operators $\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle$ which can obtained from $Z_{CFT}(J)$ by taking derivatives $\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle =$

[56]For a discussion of the partition functions and AdS boundaries, see Ed Witten "Anti de Sitter space and holography" hep-th/9802150.

[57]See hep-th/9802109.

$\frac{\delta}{\delta J_1}....\frac{\delta}{\delta J_n}Z_{CFT}(J)|_{J=0}$. On the LHS one has an on-shell theory (low energy string theory on AdS) which is obtained by solving the field equations for all fields in AdS. This means that the boundary conditions of all these fields must be specified, here as $\bar{\phi} = J$. This way it is possible to set the two partition functions equal to each other[58].

Higher spin theories:

Finally we comment on the case $R \ll l_s$ in a different limit, namely keeping R fixed while letting $R/l_s \to 0$: This case is much less understood than the previous one since it is really very stringy and hard to express in field theory terms. Here we are concerned with small λ which means that α' is large and that in the derivative expansion all terms start to be equally important. However, in the limit $\alpha' \to \infty$, the tension T goes to zero and all massive string states become **massless**. The theory is strongly believed to become a field theory called **higher spin theory**. Such theories have been developed on their own right and it is known that they must contain massless gauge fields for all values of the spin from one to infinity, possibly also with scalar fields. These higher spin theories are also believed to exist on AdS spacetimes with a CFT boundary.

[58]Clearly boundary conditions imposed on fields in AdS will play a key role here. A nice introduction to this subject can be found in Marolf and Ross, hep/th-0606113.

15 Topics for small projects

This section is divided into two parts:
1. Topics based on problems in Part 2 of BZ's book.
2. Topics for slightly more advanced projects going beyond the BZ book.

15.1 "Small projects" based on chapters in Part II of BZ

Chapter 15: D-branes and gauge fields
Project on orientifolds: Solve problem 15.4 in BZ.
Reading instructions: The mathematics in this chapter is rather easy except perhaps for the discussion of how gauge groups and gauge fields arise. Try to get a visual picture of the various open strings connecting D-branes of different kinds and how the vector states build up non-abelian gauge fields, i.e., Yang-Mills theories, on the branes. This property of the open string is crucial for the rest of the course.

Chapter 16: String charge and D-brane electric and magnetic charges
Project on string charges and Kalb-Ramond fields: Solve problems 16.3 and 16.4 in BZ.
Reading instructions: The string charge is discussed in detail in this chapter but D-brane charges are not. This latter topic will however be expanded upon in the lecture!

Chapter 17: T-duality of closed strings
Project on dualities in T^2 compactifications: Solve problems 17.4, 17.5 and 26.3 in BZ.
Reading instructions: The message of this chapter should be fairly clear namely that circle compactifications related by T-duality, i.e., by letting the radius $R \to \frac{\alpha'}{R}$, are physically equivalent! One important example is provided by type IIA and IIB superstrings (why? See chapter 18!).

Chapter 18: T-duality of open strings
Project on effects of T-duality: Solve problems 18.2, 18.5 and 18.6 in BZ.
Reading instructions: The effects of T-duality is now more complicated since Dp-branes are affected (p can change!). Explain in detail how this can happen and how it relates to the T-duality connection between type IIA and IIB string theories. Sections 18.2, 18.3 and 18.4 contain a lot of nice stuff on U(1) gauge theory and Wilson loops. However, I will only require that you know the bottom line of this discussion namely that T-duality leads to D(p-1) branes at fixed position on the T-duality circle and the consequences of this fact mentioned at the end of sect. 18.4 in connection with eqs 18.56, 18.57 and 18.58. The derivation of these results are part of the project problems in this chapter!

Chapter 19: EM fields on D-branes
Project on moving strings in electric fields: Solve problems 19.2 in BZ.
Reading instructions: Sections 19.1 and 19.2 deal with electric fields on D-branes and con-

tain all that we need from this chapter. In section 19.3 the analysis is repeated for magnetic fields. It is very similar but a bit more complicated.

Chapter 20: Born-Infeld dynamics
Project on Born-Infeld and strings ending on D-branes : Solve problems 20.6 and 20.7 in BZ.
Reading instructions: You can skip the magnetic discussion on page 444 which is based on section 19.3.

Chapter 21: String theory and particle physics
Project : Solve problem 21.3 in BZ. (Requires a bit more work than the other Part 2 problems.)
Reading instructions: Read sections 21.1 - 21.3 carefully and try to understand what goes on in section 21.4. Sections 21.5 and 21.6 (you can skip pages 484 (from eq 21.70) to 489 (up to Quick calc 21.20)) are very important for a basic understanding of the current landscape research a topic referred to as "the swampland conjectures".

Chapter 22 Black holes: Not part of the course.

Chapter 23: AdS/CFT
Project on AdS/CFT: Solve problem 23.x in BZ (decided together with the student).
Reading instructions: This is a rather difficult chapter and is included to give the student a rough idea what AdS/CFT is and what it can be used for. The main point is the duality between strongly and weakly interacting theories in the bulk and on the boundary of AdS.

Chapter 24: Covariant quantisation
Project: Solve problem 24.2 in BZ.
Reading instructions: This chapter makes more sense if (the last) section 24.6 is studied first!

Chapter 25 String interactions and Riemann surfaces: Not part of the course.

Chapter 26 Loop amplitudes: Not part of the course.

15.2 Advanced projects

The OPE, i.e., the "operator product expansion", is something that is familiar from any basic QFT course (maybe without being called OPE) which takes a particular simple form in CFT in two dimensions for instance on the string world-sheet. Consider any operator in string theory Wick rotated to Euclidean signature. Then τ and σ can be used to form a complex coordinate $z = e^{\tau - i\sigma}$ and we are interested in computing the product of two normal ordered operators, e.g., two stress tensors, at the two points z and w. The OPE is obtained by writing the product in over-all normal ordered form (as when computing the contraction in QFT) followed by a Taylor expansion of the z dependence around the point w assuming that $|z| > |w|$. The final answer of the OPE is usually given by writing out explicitly the singular terms as $z \to w$ which contain (almost) all the interesting information. This is rather easily done in most cases and it provides an amazingly efficient method to obtain a number of important results. In this set of advanced problems this method should be used in problems 1 - 4 and 10 below.

1. Deriving the Virasoro algebra using CFT

2. Scattering amplitudes: the 4-tachyon vertex

1) Derive the form the closed string coordinates $X^\mu(\tau, \sigma)$ take after Wick rotation done by replacing τ by $-i\tau$. Write the answer $X^\mu(z, \bar{z})$ as a power series in $z = e^{\tau - i\sigma}$ and $\bar{z} = e^{\tau + i\sigma}$.

2) Use the commutation relations for the zero modes x^μ, p^μ and the oscillators α_n^μ and $\bar{\alpha}_n^\mu$ to compute the contraction of $X^\mu(z, \bar{z}) X^\mu(w, \bar{w})$ by just writing the product in over-all normal ordered form as usually done for scalar fields in any QFT course. The answer is the contraction, or propagator, of $X^\mu(z, \bar{z})$. Note that to do the infinite sum one must assume that $|z| > |w|$. Hint: Do the split $X^\mu(\tau, \sigma) = X^{\mu(+)}(\tau, \sigma) + X^{\mu(-)}(\tau, \sigma)$ where the $(+)$ part contains the momentum operator p^μ and all the annihilation operators while the $(-)$ part contains position operator x^μ and all the creation operators. Then the vacuum is defined by $X^{\mu(+)}(\tau, \sigma)|0\rangle_p = 0$ where $|0\rangle_p$ is the zero momentum eigenstate as in ordinary QM times all the harmonic oscillator ground-states.

3) Next consider the tachyon vertex operator $V(z, \bar{z}) :=: e^{ik_\mu X^\mu(z, \bar{z})}$: where k^μ is the momentum (not an operator) of the tachyon. The normal ordered expression is by definition the product of two exponentials the first one containing $X^{\mu(+)}(\tau, \sigma)$ and the second annihilation $X^{\mu(-)}(\tau, \sigma)$ just as in scalar QFT. Now the task is to compute the product of two such normal ordered vertex operators which is done by writing the product in over-all normal ordered form using the Baker-Hausdorff (BH) formula $e^A e^B = e^{[A,B]} e^B e^A$ applied to the exponentials that are not automatically in normal ordered form. This form of the BH formula is valid when $[A, B]$ commutes with both A and B.

4) Generalise the result in 3) to the product of four vertex operators and take the vacuum expectation value of the answer between ${}_p\langle 0|$ and $|0\rangle_p$. You will need the fact that a momentum eigenstate can be written $|k\rangle_p = e^{ik_\mu x^\mu}|0\rangle_p$.

3. OPE and conformal dimensions

Compute the OPE between the stress tensor $T(z)$ and a vertex operator $V(z,\bar{z})$ in the following cases:

1) $V_k^T(z,\bar{z}) =: e^{ik_\mu X^\mu(z,\bar{z})}:$

2) $V_k^g(z,\bar{z}) =: \epsilon_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ik_\rho X^\rho(z,\bar{z})}:$

3) What can we say about the mass of the particles in space-time associated to these vertex operators if we require that $(h,\bar{h}) = (1,1)$?

4) Find the physical states that correspond to the operators in a) and b).

4. OPE in the bosonic string ghost sector and dimension of space-time

The propagator for the (anti-commuting) b, c system is obtained from their operator product

$$b(z)c(w) =: b(z)c(w): + \frac{1}{z-w}, \text{ for } |z| > |w|. \tag{15.1}$$

The expression for the ghost stress tensor reads

$$T(z) =: (\partial b(z))c(z): -2\partial(: b(z)c(z):). \tag{15.2}$$

1) Compute the OPE between two such stress tensors. This is best done using Wick's theorem.An OPE is what results from an operator product if the answer is expanded in a power series close the pole.

2) Interpret the result in terms of the dimension of space-time required to keep the conformal symmetry of the world-sheet theory intact (i.e., the conformal anomaly should vanish) at the quantum level.

5. Unitary representations of the Virasoro algebra

Show that the Virasoro algebra has unitary representations only for

1) $c > 0$ (Note: strictly larger than zero)

2) $h \geq 0$ where h is the eigenvalue of L_0

6. Weyl-rescalings and the duality between M-theory the type IIA string

1) Derive the formula relating the Ricci tensors for two metric fields related by a Weyl-rescaling.

2) Then compactify the bosonic part of the supergravity M-theory Lagrangian and in $D = 11$ to $D = 10$.

3) Use the Weyl-rescaling formula to relate the obtained result to both the Einstein frame and string frame formulations.

4) Show that strong coupling in $D = 10$ leads to the "opening up" of the eleventh dimension.

7. Kaluza-Klein compactification: d=11 to d=4 on S^7

1) Derive the bosonic field equations in M-theory from the Lagrangian (given).

2) Show that assuming the background geometry to be a product of 4d Minkowski and a round S^7 manifold solves the field equations. Hint: Assume that the background value of the 4-form is $< H_{\mu\nu\rho\sigma} >= 3m^2\epsilon_{\mu\nu\rho\sigma}$ where the epsilon is a tensor, not a density.

8. AdS/CFT with scalar fields in the AdS bulk

Consider an uncharged scalar field of mass m in a D-dimensional anti-de Sitter (AdS_D) spacetime.

1) Derive the behaviour of the scalar field at the boundary of AdS_D, i.e., at radial infinity. The answer should be in terms of $\Delta_\pm = \frac{d}{2} \pm \sqrt{(\frac{d}{2})^2 + m^2L^2}$ where L is the AdS scale parameter.

2) What is the meaning of the two answers $\Delta_\pm$?

3) What does "alternative" boundary conditions mean and how can one obtain these from the "standard" ones?

9. Superstrings and supersymmetry on the world-sheet

In this problem we develop the superstring world-sheet theory in two steps. To get fermions in the spacetime spectrum of a string theory it must contain fermions on the world sheet. In the so called NSR (Neveu-Schwarz-Ramond) formalism one introduces world-sheet fermions ψ^μ which are spacetime vectors like the bosonic string X^μ but contrary to the these latter ones the ψ^μ are also world-sheet spinors having two complex components. The spinor index will not be written explicitly unless absolutely necessary. To eliminate the effect of the $\mu = 0$ when quantising X^μ and ψ^μ (leading to negative norm states in Hilbert space) we need local symmetries on the world-sheet, coordinate invariance in τ, σ for X^0 and local supersymmetry for ψ^0.

1) As a first step show that the action for the super-pair X^μ, ψ^μ on a flat world-sheet

$$S[X^\mu, \psi^\mu] = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma (\tfrac{1}{2}\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} + \tfrac{i}{2}\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu}), \tag{15.3}$$

has a global supersymmetry (dropping boundary terms) under

$$\delta X^\mu = i\bar{\epsilon}\psi^\mu, \ \ \delta\psi^\mu = \rho^\alpha \epsilon \partial_\alpha X^\mu, \tag{15.4}$$

where ϵ is anti-commuting constant spinor parameter and the two-dimensional Dirac matrices ρ^α satisfy $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$ and can be chosen as follows in terms of usual Pauli matrices

$$\rho^0 = i\sigma^2, \ \ \rho^1 = -\sigma^1, \Rightarrow \bar{\rho} := \rho^0\rho^1 = -\sigma^3. \tag{15.5}$$

The Dirac conjugate above is defined as usual $\bar{\epsilon} := \epsilon\rho^0$ etc. Note that for this supersymmetry to work the fermions must be subjected to a constraint. What is this constraint?

2) The action with the required local supersymmetry is more complicated written in Polyakov form with a bosonic part involving the independent metric $h_{\alpha\beta}$ plus fermionic terms for ψ^μ and the superpartner of $h_{\alpha\beta}$ (with spin=2), the so called Rarita-Schwinger field χ_α (with spin=3/2):

$$\begin{aligned} S[X^\mu, \psi^\mu, h_{\alpha\beta}, \chi_\alpha] = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-h} (&\tfrac{1}{2}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} + \tfrac{i}{2}\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu} \\ &- \tfrac{i}{2}\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu (\partial_\beta X^\nu - \tfrac{i}{4}\bar{\chi}_\beta \psi^\nu)\eta_{\mu\nu}). \end{aligned} \tag{15.6}$$

A new aspect of this action is that it contains Dirac matrices which in a general coordinate invariant formulation requires "viel-bein" fields since Dirac matrices are only defined in flat space, the tangent space at each point. Thus $\rho^\alpha := \rho^a e_a{}^\alpha$ where ρ^a are written in tangent space with a flat index, now denoted a, and the "zwei-bein" $e_a{}^\alpha$ satisfies $\eta^{ab} e_a{}^\alpha e_b{}^\beta = h^{\alpha\beta}$. Instead of proving the local supersymmetry of this action the task here is to derive the constraints from it: one is a new version of the constraint familiar from the bosonic string (related to the vanishing of the stress tensor) and one is an entirely new anti-commuting constraint coming from the Rarita-Schwinger field equation.

10. Vertex operators, OPEs and the SU(2) Kac-Moody algebra

1) Derive the operator product of the vertex operators and the current which defines the $SU(2)$ Kac-Moody algebra.
2) The OPE version of the result in 1).
3) Mode expand the result in 2) and find the usual oscillator form of the $SU(2)$ Kac-Moody algebra.

11. The Virasoro algebra from OPEs

1) Derive the operator product of two stress tensors for the holomorphic sector of the bosonic string, e.g., a compactified component of $X^\mu(z)$.
2) Get the OPE version of the result in 1) by expanding the answer close to the pole.
3) Find the usual form of the Virasoro algebra in terms of the expansion coefficients L_n of the stress tensor by a mode analysis of the formula obtained in 2).

12. The super-Virasoro algebra from OPEs

Repeat the three steps to get the Virasoro algebra in the previous problem for the holomorphic sector of the NSR superstring. This gives a graded algebra called the super-Virasoro algebra consisting of both commutators and anti-commutators for the generators L_n, F_n or L_n, G_r.