MVE080 and MMG640 Scientific Visualization

Lecture on 3D graphics

Mathematical Sciences
CHALMERS

17th November 2020

Outline

Linear Transformations — Matrices

Linear transformations on $\mathbb{R}^3$ are studied in linear algebra, and are characterised by *linearity*:

$$\begin{cases} T(\boldsymbol{x}+\boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y}), & \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3 \\ T(\lambda \boldsymbol{x}) = \lambda T(\boldsymbol{x}), & \forall \lambda \in \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^3. \end{cases}$$

Linear Transformations — Matrices

Linear transformations on $\mathbb{R}^3$ are studied in linear algebra, and are characterised by *linearity*:

$$\begin{cases} T(\boldsymbol{x}+\boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y}), & \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3 \\ T(\lambda \boldsymbol{x}) = \lambda T(\boldsymbol{x}), & \forall \lambda \in \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^3. \end{cases}$$

If $\boldsymbol{x} = (x_1, x_2, x_3)$ then $\boldsymbol{y} = (y_1, y_2, y_3) = T(\boldsymbol{x})$ can be written as a matrix multiplication:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where $\boldsymbol{A}$ is constant.

Images of the Basic Unit Vectors

Since $(1, 0, 0)$ is transformed into

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix},$$

the first column of $\boldsymbol{A}$ is the destination of $(1, 0, 0)$.

Images of the Basic Unit Vectors

Since $(1, 0, 0)$ is transformed into

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix},$$

the first column of $\boldsymbol{A}$ is the destination of $(1, 0, 0)$.
Similarly, the second and third column tell us where $(0, 1, 0)$ and $(0, 0, 1)$ go!

Some Properties

- Matrix multiplication is *not* commutative, i.e. $\boldsymbol{AB} \neq \boldsymbol{BA}$

Some Properties

- Matrix multiplication is *not* commutative, i.e. $\boldsymbol{AB} \neq \boldsymbol{BA}$
- Matrix multiplication *is* associative, i.e. $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$

Some Properties

- Matrix multiplication is *not* commutative, i.e. $\boldsymbol{AB} \neq \boldsymbol{BA}$
- Matrix multiplication *is* associative, i.e. $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
- Some (but not all) matrices are invertible, depending on whether the *determinant* $\det \boldsymbol{A}$ is zero or not

Some Properties

- Matrix multiplication is *not* commutative, i.e. $\boldsymbol{AB} \neq \boldsymbol{BA}$
- Matrix multiplication *is* associative, i.e. $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
- Some (but not all) matrices are invertible, depending on whether the *determinant* $\det \boldsymbol{A}$ is zero or not
- If $\det \boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1}$ can be found using Gaußian elimination

Some Properties

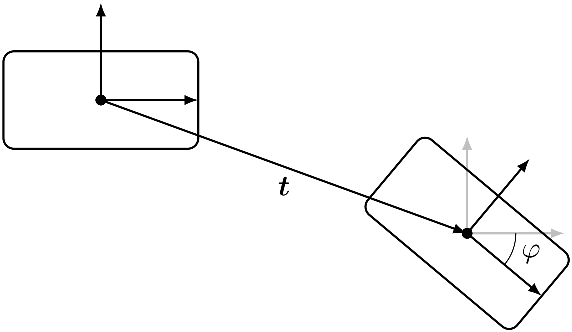
- Matrix multiplication is *not* commutative, i.e. $\boldsymbol{AB} \neq \boldsymbol{BA}$
- Matrix multiplication *is* associative, i.e. $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
- Some (but not all) matrices are invertible, depending on whether the *determinant* $\det \boldsymbol{A}$ is zero or not
- If $\det \boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1}$ can be found using Gaußian elimination
- If $\boldsymbol{A}$ is an *orthogonal* matrix, i.e. $\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A} = \boldsymbol{I}$, then $\boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathsf{T}}$

Some Properties

- Matrix multiplication is *not* commutative, i.e. $\boldsymbol{AB} \neq \boldsymbol{BA}$
- Matrix multiplication *is* associative, i.e. $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
- Some (but not all) matrices are invertible, depending on whether the *determinant* $\det \boldsymbol{A}$ is zero or not
- If $\det \boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1}$ can be found using Gaußian elimination
- If $\boldsymbol{A}$ is an *orthogonal* matrix, i.e. $\boldsymbol{A}^\intercal \boldsymbol{A} = \boldsymbol{I}$, then $\boldsymbol{A}^{-1} = \boldsymbol{A}^\intercal$
- Whatever $\boldsymbol{A}$ is, the origin is never moved, i.e. $\boldsymbol{A}\mathbf{0} = \mathbf{0}$

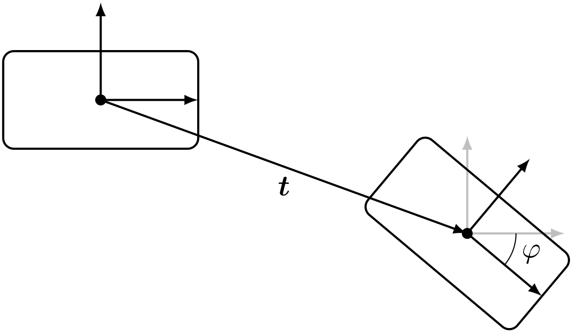
Definition of Rigid Body Motion

A rigid body motion is composed of a *rotation* and a *translation*:



Definition of Rigid Body Motion

A rigid body motion is composed of a *rotation* and a *translation*:



If combined with a *scaling*, it becomes a *similarity transformation*.

Representing Rigid Body Motions

Let $\boldsymbol{R}$ be a rotation matrix (later slides) and $\boldsymbol{t}$ be a vector. Then

$$\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$$

represents a rigid body motion.

Representing Rigid Body Motions

Let $\boldsymbol{R}$ be a rotation matrix (later slides) and $\boldsymbol{t}$ be a vector. Then

$$\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$$

represents a rigid body motion.

- Rigid body motions are not commutative

Representing Rigid Body Motions

Let $\boldsymbol{R}$ be a rotation matrix (later slides) and $\boldsymbol{t}$ be a vector. Then

$$\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$$

represents a rigid body motion.

- Rigid body motions are not commutative
- Rigid body motions are associative

Representing Rigid Body Motions

Let $\boldsymbol{R}$ be a rotation matrix (later slides) and $\boldsymbol{t}$ be a vector. Then

$$\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$$

represents a rigid body motion.

- Rigid body motions are not commutative
- Rigid body motions are associative
- Not a linear transformation — the origin is moved!

Representing Rigid Body Motions

Let $\boldsymbol{R}$ be a rotation matrix (later slides) and $\boldsymbol{t}$ be a vector. Then

$$\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$$

represents a rigid body motion.

- Rigid body motions are not commutative
- Rigid body motions are associative
- Not a linear transformation — the origin is moved!
- We will see later how to write them using only a matrix multiplication anyway!

Representing Rotations — 2D

In 2D, rotations are almost always represented using the 2×2 matrix

$$\boldsymbol{R}(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix},$$

which is an orthogonal matrix.

Representing Rotations — 2D

In 2D, rotations are almost always represented using the 2×2 matrix

$$\boldsymbol{R}(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix},$$

which is an orthogonal matrix.

To rotate a point $\boldsymbol{x} = (x, y)$ and angle φ about the origin, we do

$$\boldsymbol{y} = \boldsymbol{R}(\varphi)\boldsymbol{x} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\varphi - y\sin\varphi \\ x\sin\varphi + y\cos\varphi \end{bmatrix}.$$

Representing Rotations — 3D

In 3D, rotations around the three coordinate axes are written as

$$\boldsymbol{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix},$$

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$\boldsymbol{R}_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Representing Rotations — 3D

- Any 3D rotation can be obtained using *Tait-Bryan angles* as

$$\boldsymbol{R}(\alpha, \beta, \gamma) = \boldsymbol{R}_x(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\gamma)$$

Representing Rotations — 3D

- Any 3D rotation can be obtained using *Tait-Bryan angles* as

$$\boldsymbol{R}(\alpha, \beta, \gamma) = \boldsymbol{R}_x(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\gamma)$$

- By performing different combinations of the rotations, we get various *Euler angle* representations — no clear standard!

Representing Rotations — 3D

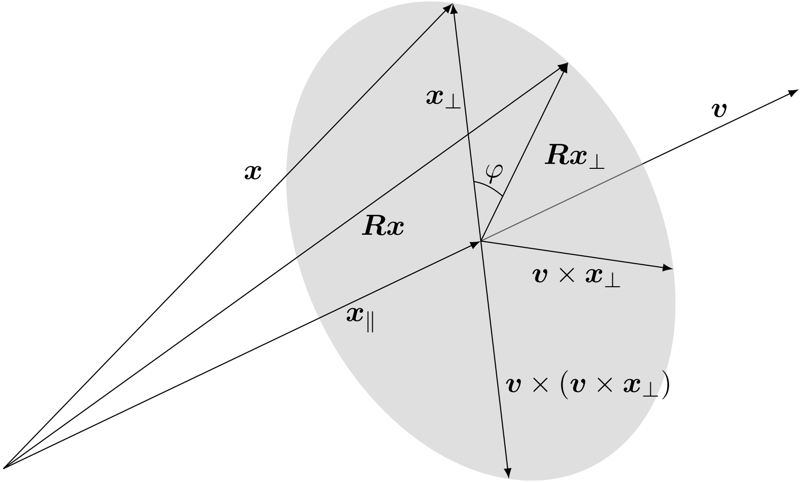
- Any 3D rotation can be obtained using *Tait-Bryan angles* as

$$\boldsymbol{R}(\alpha, \beta, \gamma) = \boldsymbol{R}_x(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\gamma)$$

- By performing different combinations of the rotations, we get various *Euler angle* representations — no clear standard!
- It is often easier to think using an *axis-angle representation*, e.g. *Rodrigues' formula*:

$$\boldsymbol{R} = \boldsymbol{I} + \sin\varphi[\boldsymbol{v}]_\times + (1 - \cos\varphi)[\boldsymbol{v}]_\times^2$$

Rodrigues' Formula



Rodrigues' Formula Proof

We have $\boldsymbol{x} = \boldsymbol{x}_{\parallel} + \boldsymbol{x}_{\perp}$, where $\boldsymbol{x}_{\parallel}$ is parallel to $\boldsymbol{v}$ (and thus does not change), and $\boldsymbol{x}_{\perp}$ is perpendicular to $\boldsymbol{v}$. Note also that $\boldsymbol{x}_{\perp}$ and $\boldsymbol{v} \times \boldsymbol{x}_{\perp}$ make up an orthogonal basis in the plane orthogonal to $\boldsymbol{v}$. It follows that

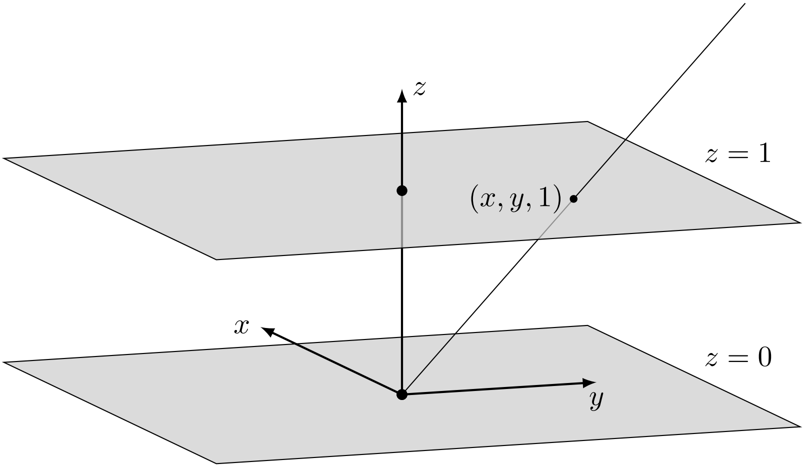
$$\begin{aligned}
\boldsymbol{R}\boldsymbol{x}_{\perp} &= \cos\varphi\, \boldsymbol{x}_{\perp} + \sin\varphi\, (\boldsymbol{v} \times \boldsymbol{x}_{\perp}) \\
&= -\cos\varphi\, (\boldsymbol{v} \times (\boldsymbol{v} \times \boldsymbol{x}_{\perp})) + \sin\varphi\, (\boldsymbol{v} \times \boldsymbol{x}_{\perp}) \\
&= -\cos\varphi\, (\boldsymbol{v} \times (\boldsymbol{v} \times \boldsymbol{x})) + \sin\varphi\, (\boldsymbol{v} \times \boldsymbol{x}).
\end{aligned}$$

Thus

$$\begin{aligned}
\boldsymbol{R}\boldsymbol{x} &= \boldsymbol{R}\boldsymbol{x}_{\parallel} + \boldsymbol{R}\boldsymbol{x}_{\perp} \\
&= (\boldsymbol{v}^{\mathsf{T}}\boldsymbol{x})\boldsymbol{v} - \cos\varphi\, (\boldsymbol{v} \times (\boldsymbol{v} \times \boldsymbol{x})) + \sin\varphi\, (\boldsymbol{v} \times \boldsymbol{x}) \\
&= \boldsymbol{x} + \sin\varphi\, (\boldsymbol{v} \times \boldsymbol{x}) + (1 - \cos\varphi)(\boldsymbol{v} \times (\boldsymbol{v} \times \boldsymbol{x})).
\end{aligned}$$

The Planar Case

Suppose we are working in the plane, and have a point (x, y).
The plane can be 'embedded' in 3D as the plane $z = 1$:



The Planar Case (contd.)

- Each point in the plane $z = 1$ corresponds to a 3D-line through the origin

Möbius, *Der barycentrische Calcul - ein neues Hülfsmittel zur analytischen Behandlung der Geometrie*, 1827.

The Planar Case (contd.)

- Each point in the plane $z = 1$ corresponds to a 3D-line through the origin
- The line through $(x, y, 1)$ includes $(\lambda x, \lambda y, \lambda)$ for any scalar λ

Möbius, *Der barycentrische Calcul - ein neues Hülfsmittel zur analytischen Behandlung der Geometrie*, 1827.

The Planar Case (contd.)

- Each point in the plane $z = 1$ corresponds to a 3D-line through the origin
- The line through $(x, y, 1)$ includes $(\lambda x, \lambda y, \lambda)$ for any scalar λ
- If $\lambda \neq 0$, we call $(\lambda x, \lambda y, \lambda)$ *homogeneous coordinates* of the point (x, y)

Möbius, *Der barycentrische Calcul - ein neues Hülfsmittel zur analytischen Behandlung der Geometrie*, 1827.

The Planar Case (contd.)

- Each point in the plane $z = 1$ corresponds to a 3D-line through the origin
- The line through $(x, y, 1)$ includes $(\lambda x, \lambda y, \lambda)$ for any scalar λ
- If $\lambda \neq 0$, we call $(\lambda x, \lambda y, \lambda)$ *homogeneous coordinates* of the point (x, y)
- Note that $(\lambda x, \lambda y, 1)$ and $(x, y, 1/\lambda)$ represent the same point

---

Möbius, *Der barycentrische Calcul - ein neues Hülfsmittel zur analytischen Behandlung der Geometrie*, 1827.

The Planar Case (contd.)

- Each point in the plane $z = 1$ corresponds to a 3D-line through the origin
- The line through $(x, y, 1)$ includes $(\lambda x, \lambda y, \lambda)$ for any scalar λ
- If $\lambda \neq 0$, we call $(\lambda x, \lambda y, \lambda)$ *homogeneous coordinates* of the point (x, y)
- Note that $(\lambda x, \lambda y, 1)$ and $(x, y, 1/\lambda)$ represent the same point
- When $\lambda \to \pm\infty$ we obtain *ideal points*, $(x, y, 0)$, infinitely far away (on the *line at infinity*)

Möbius, *Der barycentrische Calcul - ein neues Hülfsmittel zur analytischen Behandlung der Geometrie*, 1827.

The Planar Case (contd.)

- Each point in the plane $z = 1$ corresponds to a 3D-line through the origin
- The line through $(x, y, 1)$ includes $(\lambda x, \lambda y, \lambda)$ for any scalar λ
- If $\lambda \neq 0$, we call $(\lambda x, \lambda y, \lambda)$ *homogeneous coordinates* of the point (x, y)
- Note that $(\lambda x, \lambda y, 1)$ and $(x, y, 1/\lambda)$ represent the same point
- When $\lambda \to \pm\infty$ we obtain *ideal points*, $(x, y, 0)$, infinitely far away (on the *line at infinity*)
- This can be used to capture the difference between vectors and points!

Möbius, *Der barycentrische Calcul - ein neues Hülfsmittel zur analytischen Behandlung der Geometrie*, 1827.

The 3D Case

- Similarly to the 2D case, we add an extra coordinate that is equal to one, i.e. the homogeneous coordinates for (x, y, z) become $(x, y, z, 1)$ (or $(\lambda x, \lambda y, \lambda z, \lambda)$ for any $\lambda \neq 0$).

The 3D Case

- Similarly to the 2D case, we add an extra coordinate that is equal to one, i.e. the homogeneous coordinates for (x, y, z) become $(x, y, z, 1)$ (or $(\lambda x, \lambda y, \lambda z, \lambda)$ for any $\lambda \neq 0$).
- The homogeneous coordinates $(x, y, z, 0)$ represent the point infinitely far away in the direction (x, y, z)

Revisiting Rigid Body Motions

Recall that a rigid body motion consisting of the rotation $\boldsymbol{R}$ and the translation $\boldsymbol{t}$ is written as $\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$.

Revisiting Rigid Body Motions

Recall that a rigid body motion consisting of the rotation $\boldsymbol{R}$ and the translation $\boldsymbol{t}$ is written as $\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$.
As it turns out,

$$\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix},$$

so

$$\begin{bmatrix} \boldsymbol{y} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}.$$

Revisiting Rigid Body Motions

Recall that a rigid body motion consisting of the rotation $\boldsymbol{R}$ and the translation $\boldsymbol{t}$ is written as $\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$.
As it turns out,

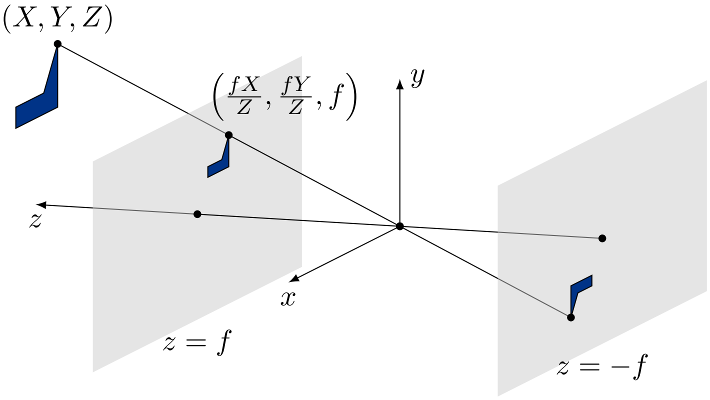
$$\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix},$$

so

$$\begin{bmatrix} \boldsymbol{y} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}.$$

If we use homogeneous coordinates, we can represent a rigid body motion as the matrix $\boldsymbol{A}$ above.

The Pinhole Perspective Camera



The Pinhole Perspective Camera (contd.)

A 3D point (X, Y, Z) is thus projected to $(fX/Z, fY/Z, f)$ in the image plane — we may omit the last coordinate:

$$(X, Y, Z) \longmapsto (fX/Z, fY/Z).$$

The Pinhole Perspective Camera (contd.)

A 3D point (X, Y, Z) is thus projected to $(fX/Z, fY/Z, f)$ in the image plane — we may omit the last coordinate:

$$(X, Y, Z) \longmapsto (fX/Z, fY/Z).$$

Using homogeneous coordinates, we can write the projection as a matrix multiplication:

$$\begin{bmatrix} fX \\ fY \\ Z \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{K}} \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}.$$

The Pinhole Perspective Camera (contd.)

- A camera positioned at $\boldsymbol{t}$ instead of the origin, and rotated a rotation $\boldsymbol{R}$, is represented by the matrix

$$\boldsymbol{P} = \boldsymbol{K}\boldsymbol{R}^{\intercal}\begin{bmatrix}\boldsymbol{I} & -\boldsymbol{t}\end{bmatrix}$$

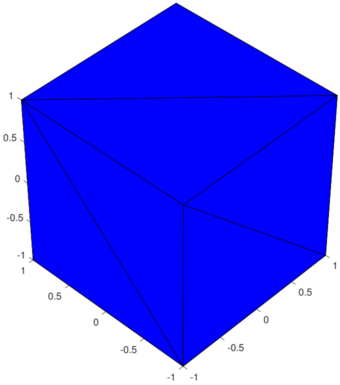
The Pinhole Perspective Camera (contd.)

- A camera positioned at $\boldsymbol{t}$ instead of the origin, and rotated a rotation $\boldsymbol{R}$, is represented by the matrix

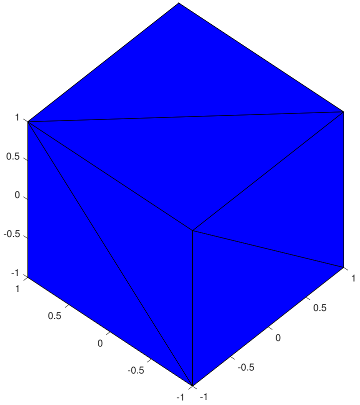
$$\boldsymbol{P} = \boldsymbol{K}\boldsymbol{R}^{\intercal}\begin{bmatrix}\boldsymbol{I} & -\boldsymbol{t}\end{bmatrix}$$

- For us, the *focal length* f is not particularly interesting most of the time — we can set it to $f = 1$ for simplicity and skip $\boldsymbol{K}$ entirely

Orthographic Cameras — Illustration

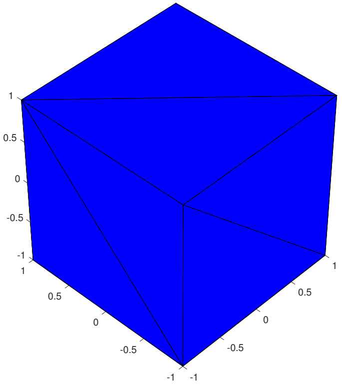


Perspective projection

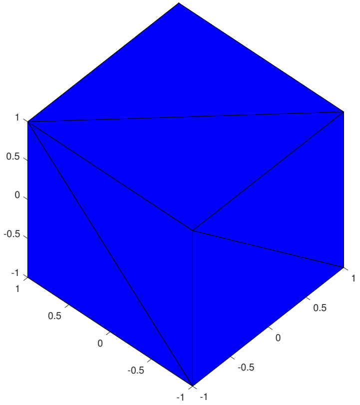


Orthographic projection

Orthographic Cameras — Illustration



Perspective projection



Orthographic projection