TMV151

Introduktion till läsvecka 4

3 videoföreläsningarna om kapitel 4
2 räkneövningar (3 fysiska, 2 i zoom)
1 frågestund på tisdag 13:15
Läs kapitel 4
Räkna övningsuppgifter i kapitel 4, i första hand (a) och (b)
Räkna problemuppgifter och gör datorövningar i Matlab eller Python

Laplacetransform



$t > 0$ tidsdomän s-domän $s \in \mathbb{C}$.

* Linjär ODE med koefficienter inhomogen transformeras till skalär ekvation i s.

konstanta

* Elektroniska nät, reglerteknik

* Laplacetransform

Antag $|f(t)| \leq Ce^{at}$, $C > 0$, $a \geq 0$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad s \in \mathbb{C}$$

$$F(s) = \mathcal{L}(f(t))(s)$$

- Integralen är konvergent om $\mathrm{Re}(s) > a$

- $\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha \mathcal{L}(f(t))(s) + \beta \mathcal{L}(g(t))(s)$

Ex: $f(t) = 1$, $|f(t)| \le C e^{at}$, $C = 1$, $a = 0$

$$F(s) = \int_0^\infty e^{-st} \cdot 1\,dt = \left[-\frac{1}{s} e^{-st} \right]_0^\infty$$

$$= -0 + \frac{1}{s} = \frac{1}{s}$$

För att $s = x + iy$, $x > 0$

$$\lim_{t\to\infty} |e^{-st}| = \lim_{t\to\infty} |e^{-(x+iy)t}| =$$

$$= \lim_{t\to\infty} |e^{-\alpha t}| \cdot |e^{-iyt}| =$$

$$= \left\{ \begin{array}{l} e^{-iyt} = \cos yt - i\sin yt \\ |e^{-iyt}| = \sqrt{\cos^2 yt + \sin^2 yt} = 1 \end{array} \right\} =$$

$$= \lim_{t\to\infty} e^{-\alpha t} = 0$$

- Skalning

Låt $g(t) = f(\alpha t)$, $\alpha > 0$

$$G_1(s) = \int_0^\infty e^{-st} f(\alpha t)\, dt =$$

$$= \left\{ \begin{array}{l} \tau = \alpha t \\ d\tau = \alpha\, dt \end{array} \right\} = \int_0^\infty e^{-s\frac{\tau}{\alpha}} f(\tau) \frac{1}{\alpha} d\tau$$

$$= \frac{1}{\alpha} \int_0^\infty e^{-\left(\frac{s}{\alpha}\right)\tau} f(\tau)\, d\tau = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$$

Exponentiell skalning

$$g(t) = e^{\alpha t} f(t)$$

$$G_1(s) = \int_0^\infty e^{-st}\, e^{\alpha t} f(t)\, dt =$$

$$= \int_0^\infty e^{-(s-\alpha)t} f(t)\,dt = F(s-\alpha)$$

Vad är Laplacetransformen av $f(t) = e^{\alpha t}$? $\mathcal{L}(e^{\alpha t})(s) = \frac{1}{s-\alpha}$

* Derivator och integraler

$$\mathcal{L}(f'(t))(s) = \int_0^\infty e^{-st} f'(t)\,dt$$

$$\overset{PI}{=} \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\cdot F(s)$$

$$\mathcal{L}(f''(t))(s) = \mathcal{L}\left(\frac{d}{dt} f'(t)\right)(s) =$$

$$= -f'(0) + s\cdot\mathcal{L}(f'(t))(s)$$

$$= -f'(0) + s(-f(0) + s\cdot F(s))$$

$$= s^2 F(s) - sf(0) - f'(0)$$

$$g(t) = \int_0^t f(\tau)\,d\tau, \quad |g(t)| \le Ce^{at}$$

$$\mathcal{L}(g(t))(s) = \int_0^\infty e^{-st} g(t)\,dt = \left[-\tfrac{1}{s} e^{-st} g(t)\right]_0^\infty$$

($f \uparrow$, $g \downarrow$, $= 0$)

$$+ \frac{1}{s}\int_0^\infty e^{-st} g'(t)\,dt = \frac{1}{s}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \frac{1}{s} F(s).$$

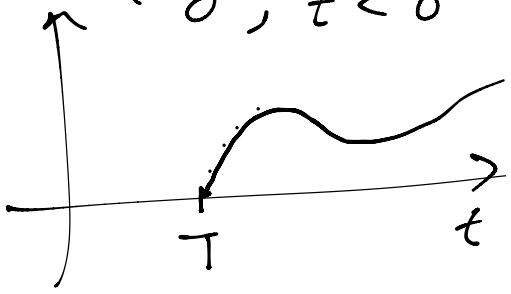
- Faltning $f * g(t) = \int_0^t f(t-u) g(u)\,du$

$f * g(t) = g * f(t)$

$\mathcal{L}(f * g(t))(s) = F(s) \cdot G(s)$

- Steg funktion $\theta(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases}$





$f(t-T)\,\theta(t-T)$ Fördröjning

$$\mathcal{L}(f(t-T)\,\theta(t-T))(s) =$$

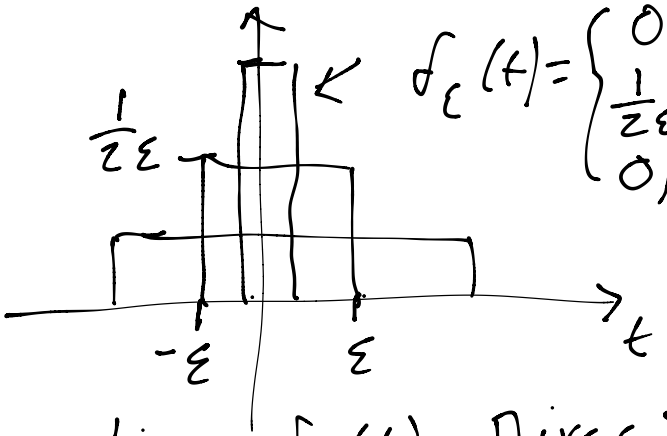
$$= \int_0^\infty e^{-st} f(t-T)\,\theta(t-T)\,dt =$$

$$= \int_T^\infty e^{-st} f(t-T)\,dt = \left\{ \begin{matrix} \tau = t-T \\ d\tau = dt \end{matrix} \right\} =$$

$$= \int_0^\infty e^{-s(\tau+T)} f(\tau)\,d\tau =$$

$$= e^{-sT} \int_0^\infty e^{-s\tau} f(\tau)\,d\tau = e^{-sT} F(s).$$

- Impuls



$$\delta_\varepsilon(t) = \begin{cases} 0, & t < -\varepsilon \\ \frac{1}{2\varepsilon}, & -\varepsilon < t < \varepsilon \\ 0, & t > \varepsilon \end{cases}$$

Låt $\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$ Diracs delta

$$\mathcal{L}(\delta(t-t_0))(s) = e^{-st_0}$$

$$\mathcal{L}(\delta(t))(s) = 1$$

* Begynnelsevärdesproblem

$$u''(t) + 4u'(t) + 4u(t) = e^{-2t}$$

$$u(0) = u'(0) = 0$$

$$\left(\mathcal{L}(t^2 e^{-2t})(s) = \frac{2}{(s+2)^3}\right)$$

1) Laplacetransformera

$$s^2 U(s) - s u(0) - u'(0) + 4(sU(s) - u(0)) + 4U(s) = \frac{1}{s+2}$$

2) Sätt in begynnelsevärden

$$u(0) = u'(0) = 0$$

$$s^2 U(s) + 4sU(s) + 4U(s) = \frac{1}{s+2}$$

3) Lös ekvationen

$$s^2 + 4s + 4 = (s+2)^2 \Rightarrow$$

$$U(s) = \frac{1}{(s+2)^3} \left| \frac{1}{s+2} = (s^2+4s+4)U(s) = \right.$$

4) Laplacetransformera tillbaka via tabell.

$\mathcal{L}^{-1}(s+2)^2 U(s)$

$$u(t) = \frac{1}{2} t^2 e^{-2t} \quad \text{Ty } U(s) = \frac{1}{(s+2)^3}.$$