TMV151

Frågestund läsvecka 4

3.20a) Visa att $u_1 = x$ löser

$$x^2 u'' + (x^2 - 2x)u' + (2-x)u = 0$$

och bestäm den allmänna lösningen

Det finns två oberoende lösningar.

$$x^2 \underbrace{u_1''}_{=0} + (x^2-2x)\underbrace{u_1'}_{=1} + (2-x)\underbrace{u_1}_{=x} =$$

$$= (x^2 - 2x) + (2-x)x = 0$$

$$u_2(x) = u_1(x)\cdot v(x) = x \cdot v(x)$$

$$u_2' = v + x\cdot v', \quad u_2'' = v' + v' + xv'' = 2v' + xv''$$

$$0 = x^2(2v' + xv'') + (x^2-2x)(v + xv') + (2-x)xv$$

$$= v''(x^3) + v'(2\cancel{x^2} + x^3 - 2\cancel{x^2}) +$$

$$+ v(\underbrace{\cancel{x^2} - 2\cancel{x} + 2\cancel{x} - \cancel{x^2}}_{=0}) = x^3(v'' + v')$$

$$\Rightarrow v'' + v' = 0 \Rightarrow w = v' \Rightarrow w' + w = 0$$

$w(x) = -Be^{-x} \Rightarrow v = A + Be^{-x}$

$\Rightarrow u_2(x) = x \cdot v = \underline{Ax} + B\underline{xe^{-x}}$, $A, B \in \mathbb{R}$.

andra oberoende lösningen till $u_2(x) = xe^{-x}$.

Den allmänna lösningen ges av

$$u(x) = Ax + Bxe^{-x}.$$

4.5c) Hitta om möjligt a och C så att $|f(t)| \leq Ce^{at}$ för $f(t) = e^{t^2}$.

Givet $C, a > 0$ antag att

$$|f(t)| \leq Ce^{at} = e^{\ln C} \cdot e^{at} = e^{at + \ln C}$$

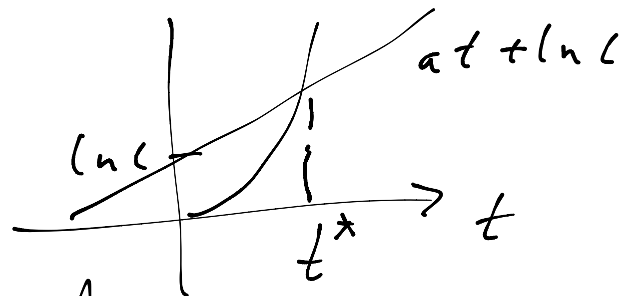
Men $t^2 > at + \ln C$ eftersom

$$t^2 = at + \ln C, \quad t^2 - at - \ln C = 0$$

$$t = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + \ln C} = t^*$$

$t^2 > at + \ln C$ om $t > t^*$.

$e^{t^2} > e^{at + \ln c}$



Alltså motsägelse $\Rightarrow$

Därför finns inga a och c

så att $|f(t)| \leq Ce^{at} \ \forall t$.

- - - - - - - - - - - -

4.7a) $f(t) = e^{4t} \cdot \cos t$ bestäm $F(s)$

- - - -

Låt $g(t) = \cos t \quad |g(t)| \leq 1 \ \forall t$

$C = 1,\ a = 0$

$e^{it} = \cos t + i \sin t$

$e^{-it} = \cos t - i \sin t$

$\frac{e^{it} + e^{-it}}{2} = \cos t \ , \ \frac{e^{it} - e^{-it}}{2i} = \sin t$

$G(s) = \int_0^\infty e^{-st} \frac{e^{it} + e^{-it}}{2} dt =$

$$= \frac{1}{2}\int_0^\infty e^{-(s-i)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(s+i)t}\,dt$$

$$= \frac{1}{2}\left[\frac{-1}{s-i}e^{-(s-i)t}\right]_0^\infty + \frac{1}{2}\left[\frac{-1}{s+i}e^{-(s+i)t}\right]_0^\infty$$

$$= \frac{1}{2}\frac{1}{s-i} + \frac{1}{2}\frac{1}{s+i} =$$

$$= \frac{1}{2}\frac{s+i}{s^2+1} + \frac{1}{2}\frac{s-i}{s^2+1} = \frac{s}{s^2+1}$$

Exponentiell skalning ger

$$\mathcal{L}(e^{\alpha t}g(t))(s) = G(s-\alpha)$$

$$\mathcal{L}(f(t))(s) = \frac{s-4}{(s-4)^2+1} = \frac{s-4}{s^2-8s+17}$$

P4.9) $u'(t) - 2u(t) + \int_0^t u(s)\,ds = e^{+t}$

$u(0) = 1$

Lös begynnelsevärdesproblemet med Laplacetransform.

1) Laplace transformera

$$sU(s) - u(0) - 2U(s) + \frac{1}{s}U(s) = \frac{1}{s-1}$$

2) Sätt in begynnelsevärden $u(0)=1$

$$sU(s) \underline{-1} - 2U(s) + \frac{1}{s}U(s) = \frac{1}{s-1}$$

3) Lös ut $U(s)$

$$\underbrace{\left(s - 2 + \frac{1}{s}\right)}_{\frac{s^2-2s+1}{s} = \frac{(s-1)^2}{s}} U(s) = \frac{1}{s-1} + 1 = \frac{1}{s-1} + \frac{s-1}{s-1}$$

$$= \frac{s}{s-1}$$

$$U(s) = \frac{s^2}{(s-1)^3}$$

4) Laplacetransformera tillbaka

$$\frac{s^2}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

$$s^2 = A(s-1)^2 + \underline{B(s-1)} + C$$

$$= s^2(A) + s(-2A + B) + 1(A - B + C)$$

$A=1 \quad -2A + B = 0 \Rightarrow B = 2 \Rightarrow C = 1$

$$U(s) = \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$u(t) = e^t + 2te^t + \frac{1}{2}t^2 e^t$$

$$\left(\mathcal{L}(e^{-t})(s) = \int_0^\infty e^{-st} e^{-t}\,dt = \int_0^\infty e^{-(s+1)t}\,dt \right.$$

$$\left. = \left[\frac{-1}{s+1} e^{-(s+1)t} \right]_0^\infty = \frac{1}{s+1}. \right)$$

4.16c) $$h(t) = f * g(t) = \int_0^t f(t-u)\,g(u)\,du$$

$$H(s) = F(s) \cdot G(s)$$

Lát $$H(s) = \frac{1}{s^2-4} = \underbrace{\frac{1}{s-2}}_{F(s)} \cdot \underbrace{\frac{1}{s+2}}_{G(s)}$$

$$f(t) = e^{2t} \quad g(t) = e^{-2t}$$

$$h(t) = \int_0^t e^{2(t-u)} e^{-2u}\,du =$$

$$= e^{2t} \int_0^t e^{-4u}\,du = e^{2t} \left[\frac{1}{-4} e^{-4u} \right]_0^t$$

$$= e^{2t} \left[\frac{1}{-4} e^{-4t} + \frac{1}{4} \right] = \frac{1}{4}\left(e^{2t} - e^{-2t}\right).$$

Alt. $H(s) = \frac{1}{s-2} \cdot \frac{1}{s+2} = \frac{A}{s-2} + \frac{B}{s+2}$

$$\frac{(s+2)A}{s^2-4} + \frac{(s-2)B}{s^2-4} = \frac{1}{s^2-4}$$

$$1 = (s+2)A + (s-2)B = s(A+B) + 1(2A-2B)$$

$A+B=0 \quad B=-A, \; 1 = 2A-2B = 4A, \; A = \frac{1}{4}$

$B = -\frac{1}{4}$

$H(s) = \frac{1/4}{s-2} - \frac{1/4}{s+2}, \; h(t) = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$