Lecture_22

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Options and Mathematics: Lecture 22

December 9, 2020

Exercises

Exercise 6.6

Let $\{W(t)\}_{t\geq 0}$ be a Brownian motion. Show that

$$\mathrm{Cov}[W(s), W(t)] = \min(s,t), \quad \text{for all } s,t \geq 0.$$

(Solution can be found in the book)

SOLUTION: RECALL THAT $\mathrm{COV}(X,Y) = E[XY] - E[X]E[Y]$

ASSUME $s \geq t$ (THE SOLUTION IS SIMILAR FOR $s \leq t$)

$\mathrm{Cov}[W(s), W(t)] = E[W(s)W(t)] - E[W(s)]E[W(t)] =$

{ SINCE $W(t) \in N(0,t)$ THEN $E[W(s)] = E[W(t)] = 0$ }

$= E[W(s)W(t)] = E[(W(s) - W(t))W(t)] + E[W(t)^2]$

$= E[(W(s) - W(t))(W(t) - \underbrace{W(0)}_{=0})] + (E[W(t)^2] - \underbrace{E[W(t)]^2}_{=0})$

$= \underbrace{E[W(s) - W(t)]}_{0} \underbrace{E[W(t)]}_{0} + \mathrm{VAR}[W(t)] = t = \min(s,t)$

WE ASSUMED $s \geq t$

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$= \mathbb{E}[W(s) - W(t)]\,\mathbb{E}[W(t)] + \mathrm{Var}[W(t)] = t = \min(s,t)$

$0 \qquad 0$

WE ASSUMED $s > t$

Exercise 6.7

Let $\{W(t)\}_{t\geq 0}$ be a $\mathbb{P}$-Brownian motion and $T > 0$. Given a differentiable function $\theta : (0, \infty) \to \mathbb{R}$, define

$$Z_\theta = \exp\left(-\theta(T)W(T) + \int_0^T \theta'(s)W(s)\,ds - \frac{1}{2}\int_0^T \theta^2(s)\,ds\right).$$

Show that $\mathbb{P}_\theta(A) = \mathbb{E}[Z_\theta \mathbb{I}_A]$ is a probability measure equivalent to $\mathbb{P}$.

HINT: You need Theorem 6.6:

Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let

$$X(t) = g(t)W(t) - \int_0^t g'(s)W(s)\,ds.$$

(REPLACE $g(t) = \theta(t)$ AND $t = T$)

Then

$$X(t) \in \mathcal{N}(0, \Delta(t)), \quad \Delta(t) = \int_0^t g(s)^2\,ds.$$

IF WE TAKE $\theta(t) = \theta$ CONSTANT THEN WE GO BACK TO THIS CASE

SOLUTION: RECALL THAT IF $Z_\theta = e^{-\theta W(T) - \frac{1}{2}\theta^2 T}$, $\theta \in \mathbb{R}$, THEN $\mathbb{P}_\theta(A) = \mathbb{E}[Z_\theta \mathbb{I}_A]$ IS, BY RADON-NYKODYM THEOREM, A PROBABILITY EQUIVALENT TO $\mathbb{P}$. (GIRSANOV'S PROBABILITY)

TO SOLVE THIS EXERCISE WE HAVE TO SHOW THAT ALSO THE NEW Z_θ SATISFIES $Z_\theta > 0$, $\mathbb{E}[Z_\theta] = 1$.

CLEARLY $Z_\theta > 0$; MOREOVER:

$$\mathbb{E}[Z_\theta] = e^{-\frac{1}{2}\int_0^T \theta(s)^2 ds}\,\mathbb{E}\left[\underbrace{e^{-\theta(T)W(T) + \int_0^T \theta'(s)W(s)\,ds}}_{e^{-X(T)},\; X(T) \in \mathcal{N}(0, \Delta(T))}\right]$$

WHERE $\Delta(T) = \int_0^T \theta(s)^2 ds$. HENCE

$$\mathbb{E}[Z_\theta] = e^{-\frac{1}{2}\int_0^T \theta(s)^2 ds} \int_{\mathbb{R}} e^{-x} f_{X(T)}(x)\,dx = e^{-\frac{1}{2}\int_0^T \theta(s)^2 ds} \int_{\mathbb{R}} e^{-\frac{x^2}{2\Delta(T)} - x} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{-\frac{1}{2}\underbrace{\int_0^T \theta(s)^2 ds}_{\Delta(T)}} \int_{\mathbb{R}} e^{-\frac{1}{2\Delta(T)}\left(x^2+2x\Delta(T)+\Delta(T)^2\right)+\frac{\Delta(T)}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= \underbrace{e^{-\frac{1}{2}\Delta(T)}\, e^{\frac{1}{2}\Delta(T)}}_{=1} \int_{\mathbb{R}} \underbrace{e^{-\frac{1}{2\Delta(T)}(x+\Delta(T))^2}}\frac{dx}{\sqrt{2\pi}} = 1$$

DENSITY OF A RV IN $N(-\Delta(T), \Delta(T))$

Exercise 6.9

Suppose that at time $t = 0$ it is assumed that the stock price is described by a geometric Brownian motion

IN $\mathbb{P}$: $\boxed{S(t) = S(0)e^{\alpha t+\sigma W(t)}}$

IN $\mathbb{P}_Q$: $S(t) = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W^{(Q)}(t)}$

THERE IS NO α HERE!

THE VALUE OF OPTIONS ON THE STOCK DOES NOT ~~DEPEND~~ DEPEND ON α. HOWEVER IT DEPENDS ON σ!!

in the interval $[0, T]$. Given any arbitrary subinterval $[t_0, t] \subset [0, T]$ with length $\tau = t - t_0$, define the random variable

$$\sigma_\tau^2(t) = \frac{1}{h(n-1)} \sum_{i=1}^{n} (\widehat{R}_i - \widehat{R})^2,$$

t_0 t T

$h = \frac{t-t_0}{n}$

where $t_0 < t_1 < t_2 < \cdots < t_n = t$ is a partition of $[t_0, t]$ with $h = t_i - t_{i-1}$ and $\widehat{R}_i$, $\widehat{R}$ are given by

RANDOM VARIABLES

$$\widehat{R}_i = \log S(t_i) - \log S(t_{i-1}) = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right), \quad i = 1, \ldots, n.$$

LOG-RETURN ON $[t_{i-1}, t_i]$

$$\widehat{R}(t) = \frac{1}{n}\sum_{i=1}^{n} \widehat{R}_i = \frac{1}{n}\log\left(\frac{S(t)}{S(t_0)}\right).$$

MEAN OF LOG RETURN

.Show that

$$\mathbb{E}[\sigma_\tau(t)^2] = \sigma^2.$$

In other words, σ^2 is the expected value of the τ-historical variance at any time $t \in [0, T]$.

(Solution can be found in the book)

$$C(T, x, K, T) = x\,\Phi(d_{(+)}) - K e^{-r\tau}\,\Phi(d_{(-)})$$

$$d_{(\pm)} = \frac{\log\frac{x}{K} + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$\tau = T - t$$

Exercise 6.12 DO IT YOURSELF

Let $C(t, x, K, T)$ be the Black-Scholes pricing function of the call with strike K and maturity T.

Prove that

$$\lim_{\sigma \to 0^+} C(t, x, K, T) = (x - Ke^{-r\tau})_+, \quad \lim_{\sigma \to \infty} C(t, x, K, T) = x.$$

Compute also the following limits:

$$\lim_{K \to 0^+} C(t, x, K, T), \quad \lim_{K \to +\infty} C(t, x, K, T), \quad \lim_{\tau \to +\infty} C(t, x, K, T), \quad \lim_{x \to 0^+} C(t, x, K, T)$$

and show that $C(t, x, K, T)$ is asymptotic to $x - Ke^{-r\tau}$ as $x \to \infty$. Compute the same limits for put options.

(Solution can be found in the book)

HINT: REMEMBER THAT $\Phi(z) \longrightarrow 1$ AS $z \to \infty$

AND $\Phi(z) \longrightarrow 0$ $z \to -\infty$

Exercise 6.13

A **binary** (or **digital**) call option with strike K and maturity T pays-off the buyer if and only if $S(T) > K$. If the pay-off is a fixed amount of cash L, then the binary call option is said to be "cash-settled", while if the pay-off is the stock itself then the option is said to be "physically settled". Compute the Black-Scholes price of the cash-settled binary call option and the number of shares on the stock in the self-financing hedging portfolio.

(Solution can be found in the book)

SOLUTION: $Y = L\,H(S(T)-K) = g(S(T))$

$$g(z) = L\,H(z-K)$$

$$\Pi_Y(t) = v(t, S(t)), \quad v(t,x) = e^{-r\tau}\int_{\mathbb{R}} g\left(x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\,y}\right) e^{-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}}$$

$$= e^{-r\tau}\int_{\mathbb{R}} L\,H\Big(\underbrace{x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\,y} - K}\Big) e^{-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}} \qquad \tau = T-t$$

$$\left\{ x e^{(r-\sigma^2/2)\tau + \sigma\sqrt{\tau}\,y} - K > 0 \text{ iff } y > \frac{\log\frac{K}{x} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = -d_{(-)} \right\}$$

$$= e^{-r\tau}\int_{-d_{(-)}}^{\infty} L\,e^{-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}} = L e^{-r\tau}\int_{-\infty}^{d_{(-)}} e^{-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}}$$

$$= L e^{-r\tau}\,\Phi(d_{(-)})$$

$$d_{(-)} = \frac{\log\frac{x}{K} + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = \frac{\log x}{\sigma\sqrt{\tau}} + (\ldots)$$

$(\ldots)$ DOESN'T DEPEND ON X

$h_S(t) = \Delta(t, S(t))$, $\Delta(t,x) = \partial_x s(t,x)$

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$\Delta(t,x) = \partial_x \left[L e^{-r\tau} \Phi(d_{(-)}) \right] = L e^{-r\tau} \Phi'(d_{(-)}) \partial_x d_{(-)}$

$= L e^{-r\tau} \varphi(d_{(-)}) \left(\frac{1}{x\sigma\sqrt{\tau}} \right) > 0$ WHERE $\varphi(z) = \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} = \Phi'(z)$

Exercise 6.15

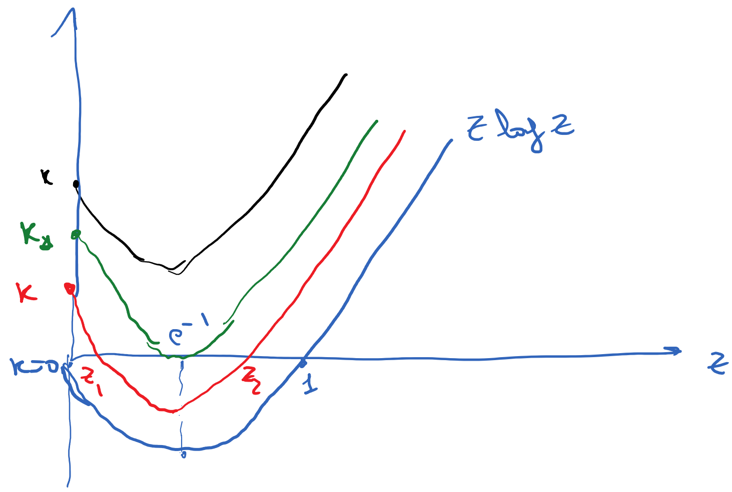
Consider the European derivative with maturity T and pay-off Y given by

$$Y = k + S(T)\log S(T),$$

where $k > 0$ is a constant. Find the Black-Scholes price of the derivative at time $t < T$ and the self-financing hedging portfolio. Find the probability that the derivative expires in the money.

(Solution can be found in the book) (DO IT YOURSELF)

HINT: $Y = g(S(T))$ $\quad g(z) = k + z\log z$



$g'(z) = \log z + 1$

$= 0 \quad z = e^{-1}$

THERE IS A VALUE OF k, SAY $k = k_*$, SUCH THAT IF $k > k_*$, THEN

$Y > 0$, AND SO

$\mathbb{P}(Y > 0) = 1$

FOR $0 < k < k_*$ THERE EXIST $z_1 < z_2$ SUCH THAT $g(z) > 0$ IFF $z < z_1$ OR $z > z_2$, HENCE

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$Y > 0 \iff S(T) < z_1$ OR $S(T) > z_2$

HENCE $\mathbb{P}(Y > 0) = \mathbb{P}(S(T) < z_1) + \mathbb{P}(S(T) > z_2)$

Solution 6.17

Let $K > 0$. A European style derivative on a stock with maturity $T > 0$ gives to its owner the right to choose between selling the stock for the price K at time T or paying the amount K at time T. Draw the pay-off function of the derivative. Compute the Black-Scholes price of the derivative. Show that there exists a value $K_* > 0$ of K such that the Black-Scholes price of the derivative is zero.

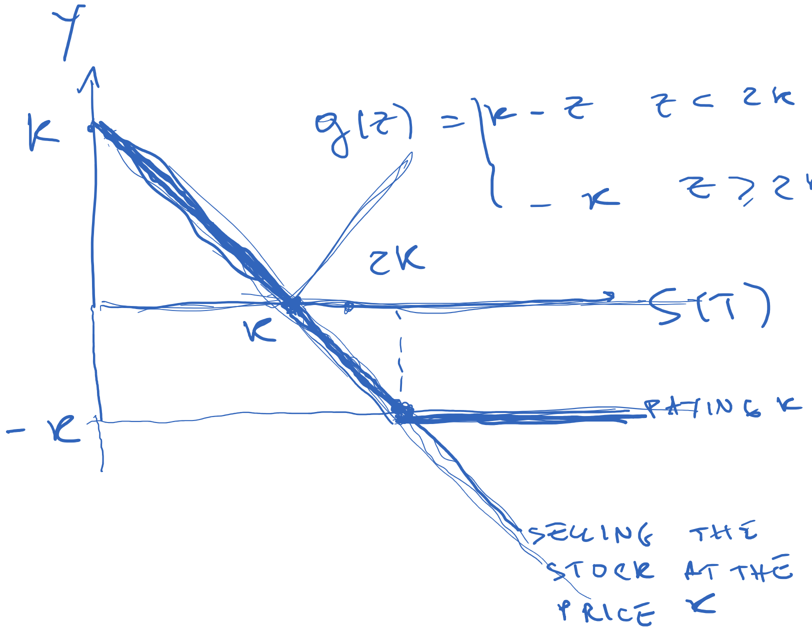
READ THE SOLUTION IN THE BOOK.

(Solution can be found in the book)

SOLUTION

FIRST DRAW THE GRAPH OF THE PAY-OFF

$$Y = g(S(T))$$



$$g(z) = \begin{cases} K - z & z < 2K \\ -K & z \geq 2K \end{cases}$$

$$Y = (K - S(T))_+ - (S(T) - K)_+ + (S(T) - 2K)_+$$

$$\Rightarrow \Pi_Y(t) = P(t, S(t), K, T) - C(t, S(t), K, T) + C(t, S(t), 2K, T) \underset{\text{PUT-CALL PARITY}}{=} Ke^{-r\tau} - S(t) + C(t, S(t), 2K, T)$$

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So, $\Pi_Y(t) = Ke^{-r\tau} - S(t) + C(t, S(t), 2K, T)$

Solution to Exercise 6.9

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$$S(t_i) = S_0 e^{\alpha t_i + \sigma W(t_i)}$$

$$\underbrace{\log S(t_i) - \log S(t_{i-1})}_{\hat{R}_i} = \alpha(t_i - t_{i-1}) + \sigma\big(W(t_i) - W(t_{i-1})\big)$$

$$R = \frac{1}{n}\sum \hat{R}_i = \frac{1}{n}\log\frac{S(t)}{S(t_0)} = \frac{1}{n}\big(\alpha(t-t_0) + \sigma(W(t) - W(t_0))\big)$$

$$\sigma_R^2(t) = \frac{1}{h(n-1)}\sum_{i=1}^{n}\Big[\cancel{\alpha(\overbrace{t_i - t_{i-1}}^{h = \frac{t-t_0}{n}})} + \sigma\big(W(t_i) - W(t_{i-1})\big) - \frac{1}{n}\big(\cancel{\alpha(t-t_0)} + \sigma(W(t) - W(t_0))\big)\Big]^2$$

WE WANT TO SHOW THAT $\mathbb{E}[\sigma_R^2(t)] = \sigma^2$

BREAK UNTIL 2.13 PM.

$$\mathbb{E}[\sigma_R^2(t)] = \frac{1}{h(n-1)}\sigma^2\,\mathbb{E}\Big[\sum_{i=1}^{n}\big(W(t_i) - W(t_{i-1})\big)^2 + \frac{1}{n^2}\big(W(t) - W(t_0)\big)^2 - \frac{2}{n}\big(W(t_i) - W(t_{i-1})\big)\big(W(t) - W(t_0)\big)\Big]$$

$$= \frac{1}{h(n-1)}\sigma^2\,\mathbb{E}\Big[\Big(\sum_{i=1}^{n}\big(W(t_i) - W(t_{i-1})\big)^2\Big) + \frac{1}{n}\big(W(t) - W(t_0)\big)^2 - \frac{2}{n}\big(W(t) - W(t_0)\big)\underbrace{\sum_{i=1}^{n} W(t_i) - W(t_{i-1})}\Big]$$

THIS SUM TELESCOPES, SO IT IS EQUAL $W(t) - W(t_0)$

$$= \frac{1}{h(n-1)}\sigma^2\,\mathbb{E}\Big[\Big(\sum_{i=1}^{n}\big(W(t_i) - W(t_{i-1})\big)^2\Big) - \frac{1}{n}\big(W(t) - W(t_0)\big)^2\Big]$$

$$= \frac{1}{h(n-1)}\sigma^2\Big(\sum_{i=1}^{n}\mathbb{E}\big[\big(W(t_i) - W(t_{i-1})\big)^2\big] - \frac{1}{n}\mathbb{E}\big[\big(W(t) - W(t_0)\big)^2\big]\Big)$$

$\mathrm{VAR}\left[W(t_i) - W(t_{i-1})\right]$

$= t_i - t_{i-1} = h$

$\mathrm{VAR}\left[W(t) - W(t_0)\right]$

$= (t - t_0)$

$$= \frac{1}{h(n-1)} \sigma^2 \left(h n - \underbrace{\frac{1}{n}(t - t_0)}_{= h} \right) = \frac{1}{\cancel{h}} \frac{\sigma^2}{\cancel{(n-1)}} \cancel{h}\,\cancel{(n-1)} = \sigma^2$$

$\blacksquare$