Options and Mathematics: Lecture 25

December 15, 2020

Introduction to bond valuation

Bonds are debt instruments issued by national governments and private companies as a way to borrow money and fund their activities.

Zero-coupon bonds

Recall that a **zero-coupon bond** (ZCB) with **face** (or **nominal**) value K and maturity $T > 0$ is a contract that promises to pay to its owner the amount K at time T in the future,

Without loss of generality we assume that $K = 1$, as one share of the ZCB with face value K is clearly equivalent to K shares of the ZCB with face value 1.

In the following we also assume that all ZCB's are issued by one given institution, so that all bonds differ merely by maturity.

A **zero-coupon bond market** (ZCB market) is a market in which the objects of trading are ZCB's with different maturities.

For modelling purposes we assume that all ZCB's in the market expire before time S (e.g., $S \approx 50$ years) and that there exists a ZCB which matures at any time $T \in (0, S)$.

Finally it is assumed that the issuing institution bears no risk of default, and thus in particular the ZCB's in the market are risk-free assets.

Let us denote by $B(t, T)$, for $t \leq T \leq S$, the value at time t of the ZCB that expires at time T. Clearly, $B(t, T) > 0$ for all $t \leq T$ and $B(T, T) = 1$.

Forward rate

The difference in value of ZCB's with different maturities is expressed through the implied forward rate of the bond.

To define this concept, suppose that at the present time t we open a portfolio that consists of

- -1 share of a ZCB with maturity $t < T$ and
- $B(t, T)/B(t, T + \delta)$ shares of a ZCB expiring at time $T + \delta$.

This investment has zero value and entails that we pay 1 at time T and receive $B(t, T)/B(t, T + \delta)$ at time $T + \delta$.

Hence our investment at the present time t is equivalent to an investment in the future time interval $[T, T + \delta]$ with (annualized) return given by

$$F_\delta(t, T) = \frac{1}{\delta} \left(B(t, T)/B(t, T + \delta) - 1 \right) = \frac{B(t, T) - B(t, T + \delta)}{\delta B(t, T + \delta)}.$$

The quantity $F_\delta(t, T)$ is also called **discretely compounded forward rate** in the interval $[T, T + \delta]$ **locked at time** t

The name is intended to emphasize that the investment return in the future interval $[T, T + \delta]$ is locked at the *present* time $t \leq T$, that is to say, we know today which interest rate has to be charged to borrow in the future time interval $[T, T + \delta]$ (if a different rate were locked today, then an arbitrage opportunity would arise).

When $\delta \to 0^+$ we obtain the **continuously compounded T-forward rate**

$$f(t,T) = \lim_{\delta\to 0^+} \frac{1}{\delta}\frac{B(t,T) - B(t,T+\delta)}{B(t,T+\delta)} = -\partial_T \log B(t,T),$$

which is the rate locked at time t to borrow at time T for an "infinitesimal" period of time. In the following we shall consider only continuously compounded rates.

The curve $T \to f(t,T)$ is called **forward rate curve** of the ZCB market.

The knowledge of the forward rate curve determines the price $B(t,T)$ of all ZCB's in the market through the formula

$$B(t,T) = \exp\left(-\int_t^T f(t,s)\,ds\right), \quad 0 \leq t \leq T \leq S,$$

The quantity

$$r(t) = f(t,t), \quad t \in [0,S]$$

is called the **(continuously compounded) spot rate** of the ZCB market at time t and represents the interest rate locked at time t to borrow instantaneously at time t (i.e., on the spot).

Example. Suppose that today (November 1st, 2017) an investor wants to sign a contract to borrow 1000000 Kr on May 1st, 2018 for a period of 6 months. There are essentially two ways in which this loan can be issued. The first way is to fix the interest rate today as the forward rate $f(t,T)$, where t =November 1st, 2017 and T =May 1st, 2018. Note that $f(t,T)$ is known at time t, as it depends on the present day price of ZCB's. The second way to issue the loan is at the spot rate $R(T)$. However the interest rate $R(T)$ is not known at time t, hence in this case the investor must wait until the first of May 2018 to know which interest rate will be charged to the loan. Of course, this second method entails a risk for both the borrower and the lender.

The spot rate can be used to define the **discount process**:

$$d(t) = \exp\left(-\int_0^t r(s)\,ds\right).$$

If t is the present time and $X(\tau)$ is the value of an asset at some given future time $\tau > t$, then the quantity

$$\frac{d(\tau)}{d(t)} X(\tau) = \exp\left(-\int_t^\tau r(s)\,ds\right) X(\tau)$$

is called the present (at time t) **discounted value** of the asset and represents the future (at time τ) value of the asset relative to the purchasing value of money at that time.

When $t = 0$, we denote the discounted value as $d(\tau)X(\tau) = X^*(\tau)$. If $r(t) = r$ is constant, we have $d(t) = e^{-rt}$, hence $X^*(\tau) = e^{-r\tau}X(\tau)$ is precisely the discounted value used for stock prices in our previous discussions.

The (continuously compounded) **yield (to maturity)** $y(t,T)$ at time t of the ZCB with maturity T is the *constant* forward rate which entails the value $B(t,T)$ of the ZCB. Hence the yield $y(t,T)$ of a ZCB is obtained by replacing $f(t,v) = y(t,T)$ in (*), i.e.,

$$B(t,T) = e^{-y(t,T)(T-t)}, \quad \text{i.e.,} \quad y(t,T) = -\frac{\log B(t,T)}{T-t}.$$

To put it in different words: Selling a ZCB for the price $B(t,T)$ at time t (i.e., borrowing $B(t,T)$ at time t) is equivalent to lock the constant forward rate $y(t,T)$ until maturity.

Coupon bonds

Let $0 < t_1 < t_2 < \cdots < t_M = T$ be a partition of the interval $[0, T]$.

A **coupon bond** with maturity T, face value 1 and coupons $c_1, c_2, \ldots, c_M \in [0, 1)$ is a contract that promises to pay the amount c_k at time t_k and the amount $1 + c_M$ at maturity $T = t_M$.

We set $c = (c_1, \ldots, c_M)$ and denote by $B_c(t, T)$ the value at time t of the bond paying the coupons $c_1, \ldots, c_M$ and maturing at time T.

Now, let $t \in [0, T]$ and $k(t) \in \{1, \ldots, M\}$ be the smallest index such that $t_{k(t)} > t$, that is to say, $t_{k(t)}$ is the first time after t at which a coupon is paid.

Holding the coupon bond at time t is clearly equivalent to holding a portfolio containing $c_{k(t)}$ shares of the ZCB expiring at time $t_{k(t)}$, $c_{k(t)+1}$ shares of the ZCB expiring at time $t_{k(t)+1}$, and so on, hence

$$B_c(t, T) = \sum_{j=k(t)}^{M-1} c_j B(t, t_j) + (1 + c_M) B(t, T),$$

the sum being zero when $k(t) = M$.

The yield of a coupon bond is defined implicitly by the equation

$$B_c(t, T) = \sum_{j=k(t)}^{M-1} c_j e^{-y_c(t,T)(t_j - t)} + (1 + c_M) e^{-y_c(t,T)(T-t)}.$$

Hence *the yield of the coupon bond is the constant spot rate used to discount the total future payments of the coupon bond.*

Most commonly the coupons are equal. Letting $c_j = c$, for all $j = 1, \ldots, M$, the formula defining the yield of coupon bonds simplifies to

$$B_c(t,T) = c\sum_{j=k(t)}^{M-1} e^{-y_c(t,T)(t_j-t)} + (1+c)e^{-y_c(t,T)(T-t)}.$$

Example.

Consider a 3 year maturity coupon bond with face value 1 which pays 2% coupon semiannually. Suppose that the bond is listed with an yield of 1%. What is the value of the bond at time zero? The coupon dates are

$$(t_1, t_2, t_3, t_4, t_5, t_6) = (1/2, 1, 3/2, 2, 5/2, 3),$$

and $c_1 = c_2 = \cdots = c_6 = 0.02$. Hence

$$\begin{aligned} B_c(0,T) = {} & 0.02e^{-0.01\cdot 1/2} + 0.02e^{-0.01\cdot 1} + 0.02e^{-0.01\cdot 3/2} + 0.02e^{-0.01\cdot 2} + 0.02e^{-0.01\cdot 5/2} \\ & + 1.02e^{-0.02\cdot 3} \approx 1.087. \end{aligned}$$

Bonds are listed in the market in terms of their yield rather than in terms of their price. The curve $T \to y_c(t,T)$ is called the **yield curve** of the ZCB market, see figures.

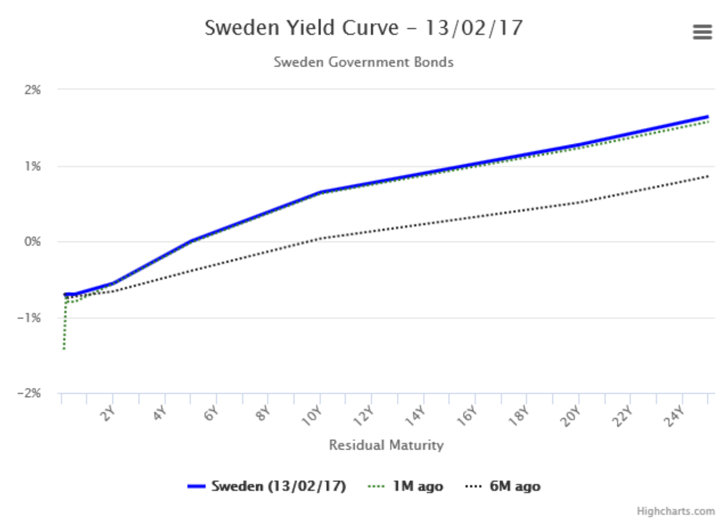

Figure 1: Yield curve for Swedish treasury bonds on February 13$^{\text{th}}$, 2017.

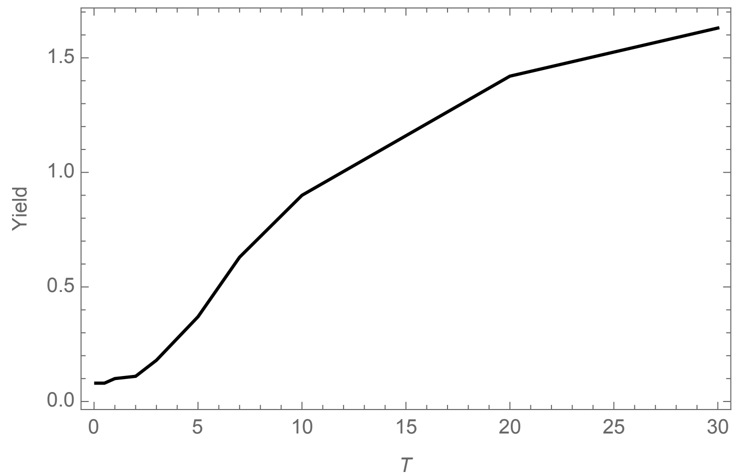

Figure 2: Yield curve for US treasury bonds on December 11$^{\text{th}}$, 2020.

The classical approach to ZCB's pricing

We conclude by describing the so-called **classical approach** to ZCB's pricing.

A cornerstone of this approach is to interpret bonds as financial derivatives on the spot rate.

As in the case of stock options, the dynamics of the underlying asset (in this case the spot rate) is prescribed *a priori* in the form of a stochastic process $\{r(t)\}_{t\in[0,S]}$.

Definition 6.10

Let $\{r(t)\}_{t\in[0,S]}$ be a stochastic process modeling the spot interest rate of the ZCB market, where we assume that that $r(0) = r_0$ is a deterministic constant. Then

$$B(0,T) = \mathbb{E}[d(T)] = \mathbb{E}[e^{-\int_0^T r(s)\,ds}], \quad 0 < T < S,$$

is called the risk-neutral value (at time $t = 0$) of the ZCB with maturity T and face value 1.

Hence the value at time $t = 0$ of the ZCB is the expected value of the discounted future payment $= 1$ of the ZCB.

Important Remark:

Note that in a purely ZCB market, one cannot define a martingale, or risk-neutral, probability, hence the expectation in the definition above is taken in the physical probability. Equivalently, in the classical approach to ZCB's pricing, the physical probability is assumed to be risk-neutral.

Exercise 6.38

In the **Vasicek model**, the risk-free rate is assumed to satisfy

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma W(t) - a\sigma \int_0^t e^{a(s-t)} W(s)\,ds, \tag{1}$$

where $\{W(t)\}_{t\geq 0}$ is a Brownian motion and $a > 0, b \in \mathbb{R}, \sigma > 0$ are constants. Show that the initial price of the ZCB with face value 1 and maturity $T > 0$ is given by

$$B(0,T) = e^{-r(0)A(T)-C(T)}, \tag{2a}$$

where

$$A(T) = \frac{1}{a}(1 - e^{-aT}), \tag{2b}$$

$$C(T) = \left(b - \frac{\sigma^2}{2a^2}\right)(T - A(T)) + \frac{\sigma^2}{4a}A(T)^2. \tag{2c}$$

Plot the yield curve at time $t = 0$ ans study numerically how it depends on the parameters a, b, σ of the model. HINT: You need Theorem 6.6.

Solution:

We have

$$B(0,T) = \mathbb{E}[\exp(-\int_0^T r(s)\,ds)].$$

For the Vasicek model the integral in the expectation is

$$\int_0^T r(s)\,ds = r(0)A(T)+bT-bA(T)+\sigma\int_0^T W(s)\,ds - a\sigma\int_0^T e^{-as}\left(\int_0^s e^{a\tau}W(\tau)d\tau\right)ds,$$

In the double integral we write $e^{-as} = -a^{-1}(de^{-as}/ds)$ and then integrate by parts in the $s-$variable to obtain

$$\sigma\int_0^T W(s)\,ds - a\sigma\int_0^T e^{-as}\left(\int_0^s e^{a\tau}W(\tau)d\tau\right)ds = \sigma\int_0^T e^{a(\tau-T)}W(\tau)\,d\tau.$$

Let $G = \int_0^T e^{a(\tau-T)}W(\tau)\,d\tau$. Choosing

$$g(t) = \frac{1-e^{a(t-T)}}{a}$$

in Theorem 6.6 we find that $G \in \mathcal{N}(0, \Delta(T))$, where

$$\Delta(T) = \int_0^T \left(\frac{1-e^{a(t-T)}}{a}\right)^2 dt.$$

It follows that

$$B(0,T) = e^{-r(0)A(T)-b(T-A(T))}\mathbb{E}[e^{-\sigma\sqrt{\Delta(T)}H}], \quad H = \frac{G}{\sqrt{\Delta(T)}} \in \mathcal{N}(0,1).$$

Using

$$\mathbb{E}[e^{-\sigma\sqrt{\Delta(T)}H}] = \int_{\mathbb{R}} e^{-\sigma\sqrt{\Delta(T)}y-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}} = e^{\frac{1}{2}\sigma^2\Delta(T)}$$

we obtain

$$B(0,T) = e^{-r(0)A(T)-b(T-A(T))+\frac{1}{2}\sigma^2\Delta(T)},$$

from which the formula in the exercise follows after some straightforward calculations.