Exam for the course “Options and Mathematics” (CTH[*MVE095*], GU[*MMG810*]) 2019/20

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REMARKS: (1) No aids permitted (2) Minor errors in the calculations will be forgiven, but remember that fractions look nicer when you simplify them!

Part I

1. Assume that the market is frictionless, arbitrage free and that the assets pay no dividend. Prove the put-call parity (max 2 points). **Solution.** See Theorem 1.2(a).

2. Consider a binomial market with parameters u, d, r, p. Show that the market admits a unique martingale probability if and only if $d < r < u$ (max 2 points). **Solution.** See Theorem 5.4

3. Give and explain the definition of Black-Scholes price of European derivatives using the risk-neutral pricing formula (max 2 points). **Solution.** See Definition 6.3.

4. Decide whether the following statements are true or false and explain your answer (max 2 points):

   (a) In a frictionless, arbitrage free market, the value of the American put converges to its intrinsic value as the risk-free rate tends to infinity.

   (b) In a frictionless, arbitrage free market, the value of the European put converges to zero as the risk-free rate tends to infinity.

Solution. (a) is true. In fact if r becomes arbitrarily large, so will do the return of the risk-free asset in any interval $[0, T]$ with $T > 0$. Hence the only case in which an American put bought at time $t = 0$ may beat the risk-free asset is when it is exercised immediately; therefore the value of the American put at $t = 0$ must equal its intrinsic value to avoid arbitrage. (b) is also true, e.g., because the value at time $t = 0$ of the European put is non-negative and bounded above by Ke^{-rT}.

Part II

1. Let $K > 0$ and $S(t) > 0$ be the price at time t of a non-dividend paying stock. A European style derivative with expiration $T > 0$ gives the obligation to either sell the stock for the price K at time T or to buy the stock for the price $2K$ at time T. Find a constant portfolio of European call/put options that replicates the derivative for $t \leq T$ (max 2 points). Assume $r \geq 0$ and show that the value of the derivative is positive at time $t = 0$ if $K < S(0)/2$ (max 2 points).

Solution. The pay-off of the derivative is $Y = \max(K - S(T), S(T) - 2K)$. As $Y = (K - S(T))_+ - (S(T) - K)_+ + 2(S(T) - 3K/2)_+$, then the derivative is replicated by a portfolio with 1 share of the put with strike K, -1 share of the call with strike K and 2 shares of the call with strike $3K/2$. Moreover by the put-call parity

$$\begin{aligned}
\Pi_Y(0) &= P(0, S(0), K, T) - C(0, S(0), K, T) + 2C(0, S(0), \frac{3}{2}K, T) \\
&= -S(0) + Ke^{-rT} + 2C(0, S(0), \frac{3}{2}K, T) \\
&= -S(0) + Ke^{-rT} + 2(C(0, S(0), \frac{3}{2}K, T) - P(0, S(0), \frac{3}{2}K, T)) + 2P(0, S(0), \frac{3}{2}K, T) \\
&= S(0) - 2Ke^{-rT} + 2P(0, S(0), \frac{3}{2}K, T) \geq S(0) - 2K,
\end{aligned}$$

by which it follows that $\Pi_Y(0) > 0$ for $K < S(0)/2$.

2. A lookback call option with maturity T and floating strike gives the right to buy the underlying stock at time T for the minimum price of the stock in the interval $[0, T]$. Consider a 3-period binomial model with parameters

$$u = \log\frac{4}{3}, \ d = \log\frac{2}{3}, \ r = 0, \ S(0) = 2, \ p = \frac{1}{2}.$$

Compute the initial price of the lookback call option on the stock with maturity $T = 3$ (max 2 points). Compute also the probability of positive return for the buyer of the derivative *conditional* to the event that the derivative expires in the money (max 2 points).

Solution. The pay-off of the derivative is $Y = S(3) - \min(S(0), S(1), S(2), S(3))$. Using the given market parameters one finds

$$\begin{aligned}
Y(u,u,u) &= \frac{74}{27}, \quad Y(u,u,d) = \frac{10}{27}, \quad Y(u,d,u) = \frac{16}{27}, \quad Y(u,d,d) = 0, \\
Y(d,u,u) &= \frac{28}{27}, \quad Y(d,u,d) = 0, \quad Y(d,d,u) = \frac{8}{27}, \quad Y(d,d,d) = 0.
\end{aligned}$$

The price of the derivative at time $t = 0$ is

$$\Pi_Y(0) = e^{-3r}[(q_u)^3\frac{74}{27} + (q_u)^2 q_d(\frac{10}{27} + \frac{16}{27} + \frac{28}{27}) + q_u(q_d)^2\frac{8}{27}].$$

Using $r = 0$ and $q_u = q_d = 1/2$, one finds $\Pi_Y(0) = \frac{17}{27}$. The probability of positive return conditional to the event that the derivative expires in the money is

$$\mathbb{P}(R > 0|Y > 0) = \frac{\mathbb{P}(\{R > 0\} \cap \{Y > 0\})}{\mathbb{P}(Y > 0)} = \frac{\mathbb{P}(R > 0)}{\mathbb{P}(Y > 0)} = \frac{p^2}{p + p^2 - p^3} = \frac{1}{1 + p - p^2} = \frac{2}{5} = 40\%$$

3. Compute the Black-Scholes price at time $t = 0$ of the European derivative with pay-off $Y = |S(T) - S(0)e^{rT}|$ and maturity T, where $S(t)$ is the price of the underlying stock (max 2 points). Show that there exists a unique value σ_* of the stock volatility such that $\Pi_Y(0) > S(0)$ if $\sigma > \sigma_*$ and $\Pi_Y(0) < S(0)$ for $\sigma < \sigma_*$ (max 2 points).

Solution. Using $|x-k| = (x-K)_+ + (K-x)_+$ we obtain

$$\Pi_Y(0) = C(0, S(0), K, T) + P(0, S(0), K, T), \quad K = S(0)e^{rT}.$$

By the Black-Scholes formula for call/put options we find

$$\Pi_Y(0) = 2S(0)[\Phi(\frac{\sigma\sqrt{T}}{2}) - \Phi(\frac{-\sigma\sqrt{T}}{2})] = 2S(0)\left(\frac{1}{\sqrt{2\pi}}\int_{-\sigma\sqrt{T}/2}^{\sigma\sqrt{T}/2} e^{-\frac{1}{2}x^2}\,dx\right)$$

Let $F(\sigma)$ be the function within round brackets. Since $F(0) = 0$, $F(\infty) = 1$ and $F'(\sigma) > 0$, there exists a unique value σ_* of σ such that $F(\sigma) > 1/2$ for $\sigma > \sigma_*$ and $F(\sigma) < 1/2$ for $\sigma < \sigma_*$.