Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMG810*]) 2019/20
For questions call the examiner at +46 (0)31 772 35 62

August 25$^{\text{th}}$, 2020

REMARKS: (1) All aids permitted, however you must work alone (2) See the course homepage for instructions on how to submit the exam.

Part I: The number of points assigned to the tasks in this part will be based on the amount of details provided in the solution. Use your own words and explain each step in the proof, in more details than in the lecture notes. Do not skip calculations! No point will be awarded by writing "just" what is written in the lecture notes.

1. Derive the formula for the self-financing replicating portfolio process of European derivatives in a binomial market (max 2 points).

2. Derive the probability density function of the stock price in Black-Scholes markets (max 2 points).

3. Give and explain the definition of Black-Scholes price at time $t = 0$ of European derivatives using the risk-neutral pricing formula (max 2 points).

4. Let $\mathcal{U}_1$ be the call stock option with strike K_1 and maturity T and $\mathcal{U}_2$ be the physically-settled digital call option on the same stock with strike K_2 and maturity T. Decide whether the following statements are true or false in an arbitrage-free market and explain your answer (max 2 points):

   (a) If $K_2 \leq K_1$, the value of $\mathcal{U}_2$ is greater or equal than the value of $\mathcal{U}_1$ for all $t < T$

   (b) If $K_2 > K_1$, the value of $\mathcal{U}_1$ is greater or equal than the value of $\mathcal{U}_2$ for all $t < T$

   Solution: (a) True, because the pay-off of $\mathcal{U}_2$ is larger or equal than the pay-off of $\mathcal{U}_1$ (b) False, because the pay-off of $\mathcal{U}_2$ is strictly lower, resp. larger, than the pay-off of $\mathcal{U}_1$ for $S(T) \in (K_1, K_2)$, resp. $S(T) > K_2$. As the price of any derivative tends to the pay-off as $t \to T$, then, by continuity, for t sufficiently close to maturity, the difference in value between $\mathcal{U}_2$ and $\mathcal{U}_1$ cannot have a definite sign.

Part II: Do not skip calculations in the exercises and write as clear as possible. If some portion of the solution is not clearly readable, it will be assumed to be wrong.

1. Find a constant portfolio consisting of European puts that replicates the European derivative with maturity T and pay-off Y depicted in the next page (max 4 points).

   Solution: -1 share of the put with strike 1, -3 shares of the put with strike 2, 1 share of the put with strike 4.

2. A European derivative with expiration $T = N$ pays the amount $Y = \log(S(T)/K)$. Find K such that the binomial price of the derivative at time $t = 0$ is zero (max. 3 points). What is the financial meaning of this value of K? (max. 1 point) HINT: Use the identity

$$\binom{N}{k} k = N \binom{N-1}{k-1}.$$

   Solution: The binomial price of the derivative at time $t = 0$ is given by

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} q^{N_u(x)} (1-q)^{N_d(x)} \log(S(T)/K),$$

   where $q = (e^r - e^d)/(e^u - e^d)$, $N_u(x)$ is the number of times the stock goes up and $N_d(x) = N - N_u(x)$ the number of times it goes down. Substituting

$$S(T) = S(0) e^{N_u(x) u + N_d(x) d}$$

   we find

$$\Pi_Y(0) = e^{-rN} \sum q^{N_u(x)} (1-q)^{N - N_u(x)} [(N_u(x) u + (N - N_u(x)) d) - \log(K/S(0))].$$

   Letting $k = N_u(x)$ and applying the binomial theorem we find

$$\begin{aligned} \Pi_Y(0) &= e^{-rN} \sum_{k=0}^{N} \binom{N}{k} q^k (1-q)^{N-k} [(ku + (N-k)d) - \log(K/S(0))] \\ &= e^{-rN} \sum_{k=0}^{N} \binom{N}{k} k\, q^k (1-q)^{N-k} (u-d) + e^{-rN} N d - e^{-rN} \log(K/S(0)). \end{aligned}$$

   Using the formula in the HINT and applying again the binomial theorem we find

$$\sum \binom{N}{k} k\, q^k (1-q)^{N-k} = N \sum_{k=1}^{N} \binom{N-1}{k-1} q^k (1-q)^{N-k} = Nq \sum_{k=0}^{N-1} \binom{N-1}{k} q^k (1-q)^{N-1-k} = Nq.$$

   Hence

$$\Pi_Y(0) = e^{-rN} [N(qu + (1-q)d) - \log(K/S(0))].$$

   Thus the value of K such that $\Pi_Y(0) = 0$ is $K = S(0) e^{N(qu + (1-q)d)} = K_*$ (3 points). Since this derivative is a forward contract, then the fair value of this derivative is zero, hence K_* is the fair value of the strike upon which the buyer and the seller should agree when they stipulate the contract (1 point).

3. Compute the Black-Scholes price at time $t < T$ of the European derivative with pay-off $Y = (S(T)^\delta - K)_+$ and maturity T, where $S(t)$ is the price of the underlying stock and $\delta > 0$ and $K > 0$ are constants (max. 3 points). Compute the limit of the derivative value as $\delta \to 0^+$ (max. 1 point).

Solution: The Black-Scholes price is given by $\Pi_Y(t) = v(t, S(t))$, where the pricing function $v(t,x)$ is computed with the integral formula

$$v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(x^\delta e^{\delta\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\delta\sqrt{\tau} y} - K \right)_+ e^{-\frac{1}{2}y^2}\, dy.$$

Using the exact same argument as for the standard call option $\delta = 1$, we obtain

$$v(t,x) = x^\delta e^{-(1-\delta)r\tau} e^{-\frac{\sigma^2}{2}\tau\delta(1-\delta)} \Phi(d_{1,\delta}) - K e^{-r\tau} \Phi(d_{2,\delta}),$$

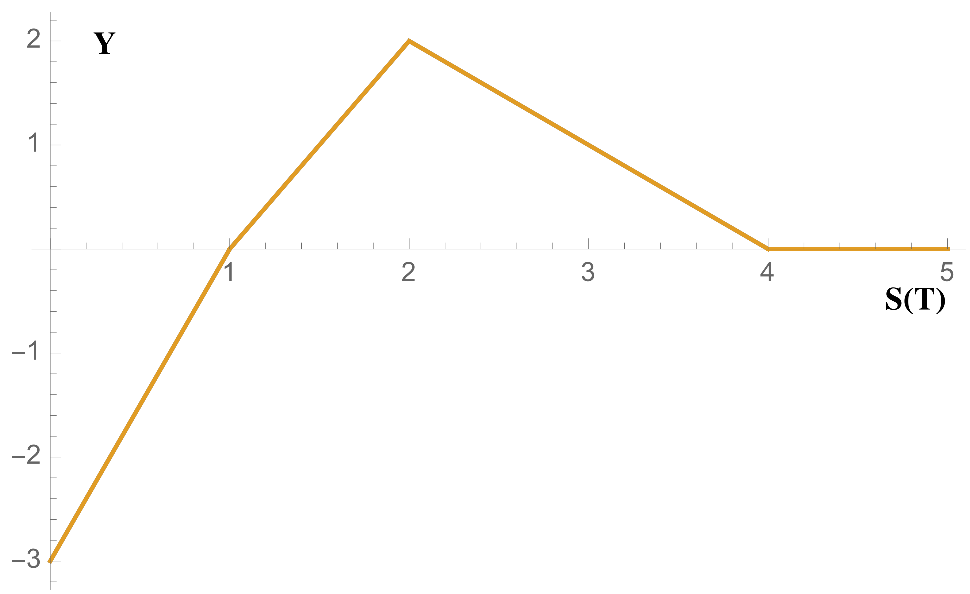
where $d_{1,\delta} = d_{2,\delta} + \delta\sigma\sqrt{\tau}$ and

$$d_{2,\delta} = \frac{\log \frac{x}{K^{1/\delta}} + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

Taking the limit $\delta \to 0^+$ we find

$$\lim_{\delta \to 0^+} \Pi_Y(t) = e^{-r\tau}(1-K)H(1-K) = e^{-r\tau}(1-K)_+,$$

where $H(x)$ is the Heaviside function. REMARK: The same result is obtained by substituting $Y = (1-K)_+$ in the risk-neutral pricing formula.



Remark: For $S(T) > 4$ the pay-off is identically zero.