Options and Mathematics: Lecture 1

November 3, 2020

Basic financial concepts

Financial assets

The course *options and mathematics* deals with the theoretical valuation of financial assets, such as

- Stocks
- Stock options
- Forward contracts
- Bonds

Exchange markets and OTC markets

Financial assets can be traded in

- Official exchange markets
- or Over The Counter (**OTC**)

Asset price

- **bid price** = maximum price that the buyer is willing to pay for the asset
- **ask price** = minimum price at which the seller is willing to sell the asset

When the **bid-ask spread** becomes zero, the exchange of the asset takes place at the corresponding price.

Notation

- $\mathcal{U} \equiv$ generic (financial) asset
- $\Pi^{\mathcal{U}}(t) \equiv$ asset price at time t

Remarks

1. Special notation for some specific assets (e.g., $S(t) \equiv$ price of a stock at time t)
2. Price are given in an unspecified unit of currency (e.g., dollars)
3. Price always refers to price per **share** of the asset
4. Any transaction in the market is subject to **transaction costs** (e.g., broker's commissions) and **transaction delays** (trading in real markets is not instantaneous).

Short-selling

An investor is said to short-sell N shares of an asset if the investor borrows the shares from a third party and sell them immediately in the market.

The reason for short-selling an asset is the expectation that the price of the asset will **decrease** in the future.

Example:

Long and short position

An investor is said to have a

- **long position** on an asset if the investor owns the asset and will therefore profit from an increase of its price.
- **short position** if the investor will profit from a decrease of its value, as it happens for instance when the investor is short-selling the asset.

Stock dividend

A stock may occasionally pay a **dividend** to its shareholders, usually in the form of a cash deposit.

- **Announcement day**≡ day when it is announced that the stock with pay the dividend D at time T in the future
- **Ex-dividend day**≡ first day before the payment date at which buying the stock does not entitle to the dividend
- **Payment day**≡ the day T at which the dividend is payed

At the ex-dividend day, the price of the stock often (but not always!) drops of roughly the same amount paid by the dividend.

Exercise 1.1[?]: Explain why it is reasonable to expect that at the ex-dividend day the price of the stock will drop by the same amount paid by the dividend..

Portfolio position and portfolio process

A **portfolio** is the collection of all asset shares owned by the investor.

Consider an agent is investing on

- a_1 shares of the asset $\mathcal{U}_1$,
- a_2 shares on $\mathcal{U}_2$,
- $\ldots$,
- a_N shares on $\mathcal{U}_N$.

The vector $\mathcal{A} = (a_1, a_2, \ldots, a_N) \in \mathbb{Z}^N$ is called a **portfolio position**.

A **positive** number of shares correspond to a **long position**, while a **negative** number of shares corresponds to a **short** position

Portfolio value at time t:

$$\boxed{V_{\mathcal{A}}(t) = \sum_{i=1}^{N} a_i \Pi^{\mathcal{U}_i}(t)}$$

- $a_i > 0$ means long position on $\mathcal{U}_i$ (portfolio value increases when price of $\mathcal{U}_i$ increases)
- $a_i < 0$ means short position on $\mathcal{U}_i$ (portfolio value increases when price of $\mathcal{U}_i$ decreases)

Remark: Portfolios can be added using the linear structure of $\mathbb{Z}^N$.

A **portfolio process** is a portfolio in which the position on the different assets changes in time.

Suppose that the investor changes the position on the assets at some times $t_1, \ldots, t_{M-1}$, where

$$0 = t_0 < t_1 < t_2 < \cdots < t_{M-1} < t_M = T;$$

We call $\{0 = t_0, t_1, \ldots, t_M = T\}$ a **partition** of the interval $[0, T]$.

Denote

- $\mathcal{A}_0 \equiv$ initial (at $t = 0$) portfolio position
- $\mathcal{A}_j \equiv$ portfolio position in the interval $(t_{j-1}, t_j]$, $j = 1, \ldots, M$

As positions hold for one instance of time only are meaningless, we assume $\mathcal{A}_0 = \mathcal{A}_1$, i.e.,

$$\boxed{\mathcal{A}_1 \textit{ is the portfolio position in the closed interval } [0, t_1]}$$

The vector $(\mathcal{A}_1, \ldots, \mathcal{A}_M)$ is called a **portfolio process**.

Denoting by a_{ij} the number of shares of the asset i in the portfolio $\mathcal{A}_j$, the value of the portfolio process at all times is given by

$$V(t) = \begin{cases} V_{\mathcal{A}_1}(t) = \sum_{i=1}^N a_{i1} \Pi^{\mathcal{U}_i}(t), & \text{for } t \in [0, t_1] \\ V_{\mathcal{A}_2}(t) = \sum_{i=1}^N a_{i2} \Pi^{\mathcal{U}_i}(t), & \text{for } t \in (t_1, t_2] \\ \vdots & \vdots \\ V_{\mathcal{A}_M}(t) = \sum_{i=1}^N a_{iM} \Pi^{\mathcal{U}_i}(t), & \text{for } t \in (t_{M-1}, t_M] \end{cases}$$

The initial value $V(0) = V_{\mathcal{A}_0} = V_{\mathcal{A}_1}(0)$ of the portfolio, when it is positive, is called the **initial wealth** of the investor.

Self-financing portfolio

A portfolio process is said to be **self-financing** if the portfolio assets pay no dividends and if no cash is ever withdrawn or infused in the portfolio.

Example: Let $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ be non-dividend paying assets in the interval $[0, T]$.

Consider a portfolio process on these assets with initial position

$$\mathcal{A}_0 = (-400, 200, 100),$$

whose value is

$$V_{\mathcal{A}_0} = -400\, \Pi^{\mathcal{U}_1}(0) + 200\, \Pi^{\mathcal{U}_2}(0) + 100\, \Pi^{\mathcal{U}_3}(0).$$

This value can be positive, zero or negative.

When $V_{\mathcal{A}_0} > 0$ we call it **initial wealth** of the investor.

The value of the portfolio process in the interval $[0, t_1]$ is

$$V(t) = -400\, \Pi^{\mathcal{U}_1}(t) + 200\, \Pi^{\mathcal{U}_2}(t) + 100\, \Pi^{\mathcal{U}_3}(t).$$

Suppose that at time $t = t_1$ the investor

- buys 500 shares of $\mathcal{U}_1$,
- sells x shares of $\mathcal{U}_2$,
- sells all the shares of $\mathcal{U}_3$.

In the interval $(t_1, t_2]$ the investor has a new portfolio which is given by

$$\mathcal{A}_2 = (100, 200 - x, 0), \quad \text{with value} \quad V(t) = 100\, \Pi^{\mathcal{U}_1}(t) + (200 - x)\, \Pi^{\mathcal{U}_2}(t)$$

Question: Can this new portfolio position be created without adding or removing cash from the portfolio?

To answer this we take the limit of $V(t)$ as $t \to t_1^+$:

$$V(t_1^+) := \lim_{t \to t_1^+} V(t) = 100\, \Pi^{\mathcal{U}_1}(t_1) + (200 - x)\, \Pi^{\mathcal{U}_2}(t_1)$$

$V(t_1^+)$ is the value of the portfolio "immediately after" changing the position at time t_1.

The difference between the value of the two portfolios immediately after and immediately before the transaction is then

$$\begin{aligned} V(t_1^+) - V(t_1) &= 100\, \Pi^{\mathcal{U}_1}(t_1) + (200 - x)\, \Pi^{\mathcal{U}_2}(t_1) \\ &\quad - (-400\, \Pi^{\mathcal{U}_1}(t_1) + 200\, \Pi^{\mathcal{U}_2}(t_1) + 100\, \Pi^{\mathcal{U}_3}(t_1)) \\ &= 500\, \Pi^{\mathcal{U}_1}(t_1) - x\, \Pi^{\mathcal{U}_2}(t_1) - 100\, \Pi^{\mathcal{U}_3}(t_1). \end{aligned}$$

If $V(t_1^+) - V(t_1)$ is positive, then the new portfolio cannot be created from the old one without infusing extra cash.

If $V(t_1^+) - V(t_1)$ is negative, then the new portfolio is less valuable than the old one, the difference being equivalent to cash withdrawn from the portfolio.

For a self-financing portfolio processes we must have $V(t_1^+) - V(t_1) = 0$, and similarly

$$V(t_j^+) - V(t_j) = 0, \text{for all } j = 1, \ldots, M-1 \quad \text{(self-financing portfolio).}$$

Thus the number x of shares of $\mathcal{U}_2$ to be sold at time t_1 in a self-financing portfolio is

$$x = \frac{500\Pi^{\mathcal{U}_1}(t_1) - 100\Pi^{\mathcal{U}_3}(t_1)}{\Pi^{\mathcal{U}_2}(t_1)}.$$

Of course, x will be an integer only in exceptional cases, which means that *perfect self-financing strategies in real markets are almost impossible.*

Portfolio generating a cash flow

If $V(t_j^+) \neq V(t_j)$, we say that at time t_j the portfolio process generates the **cash flow**

$$C(t_j) = -(V(t_j^+) - V(t_j))$$

- a positive cash flow corresponds to cash *removed* from the portfolio (causing a decrease of its value),
- a negative cash flow corresponds to cash *added* to the portfolio.

Remarks

1. The total cash flow generated by the portfolio process in the interval $[0, T]$ is $C_{\text{tot}} = \sum_{j=1}^{M-1} C(t_j)$ and can be negative, positive or zero.
2. If an asset pays a dividend D at some time $t_* \in (0, T)$, then the portfolio process generates the positive cash flow D at time t_* if the portfolio is long on the asset and the negative cash flow $-D$ if it is short on the asset
3. Constant portfolio positions are self-financing provided the assets pay no dividends.

Portfolio return

Consider a *self-financing* portfolio process opened at time $t = 0$ and closed at time $t = T > 0$.

Let $V(t)$ be the value of the portfolio at time $t \in [0, T]$.

We define

$$R(T) = V(T) - V(0) \quad \textbf{return} \text{ of the portfolio in the interval } [0, T],$$

If $R(T) > 0$ the investor makes a **profit** in the interval $[0, T]$.

If $R(T) < 0$ the investor incurs in a **loss** in the interval $[0, T]$.

When $V(0) > 0$ we define

$$R_{\text{rate}}(T) = \frac{V(T) - V(0)}{V(0)} \quad \textbf{rate of return} \text{ of the portfolio in the interval } [0, T].$$

The total cash flow C generated by a (non-self-financing) portfolio process must be included in the computation of the return of the portfolio in the interval $[0, T]$ according to the formula

$$\boxed{R(T) = V(T) - V(0) + C.}$$

Portfolio returns are commonly "annualized" by dividing the return $R(T)$ by the time T expressed in fraction of years (e.g., $T = 6$ months $= 1/2$ years).

Remark: 1 day $= 1/252$ years (markets are closed in the week-ends!)

Assets return

Consider now a portfolio that consists of a long position on one share of the asset $\mathcal{U}$ in the interval $[t, t+h]$ and assume that the asset pays no dividend in this time interval.

The annualized rate of return of this portfolio is given by

$$\boxed{R_h(t) = \frac{\Pi^{\mathcal{U}}(t+h) - \Pi^{\mathcal{U}}(t)}{h\,\Pi^{\mathcal{U}}(t)}}$$

and is also called **simply compounded** rate of return of $\mathcal{U}$.

In the limit $h \to 0^+$ we obtain the **continuously compounded** (or **instantaneous**) rate of return of the asset:

$$r(t) = \lim_{h\to 0^+} R_h(t) = \frac{1}{\Pi^{\mathcal{U}}(t)} \lim_{h\to 0^+} \frac{\Pi^{\mathcal{U}}(t+h) - \Pi^{\mathcal{U}}(t)}{h} = \frac{1}{\Pi^{\mathcal{U}}(t)} \frac{d\Pi^{\mathcal{U}}(t)}{dt}$$

that is

$$\boxed{r(t) = \frac{d \log \Pi^{\mathcal{U}}(t)}{dt}}$$

Asset returns are often computed using the logarithm of the price rather than the price itself.

For instance the quantity

$$\boxed{\widehat{R}_h(t) = \log \Pi^{\mathcal{U}}(t+h) - \log \Pi^{\mathcal{U}}(t) = \log\left(\frac{\Pi^{\mathcal{U}}(t+h)}{\Pi^{\mathcal{U}}(t)}\right)}$$

is called **log-return** of the asset $\mathcal{U}$ in the interval $[t, t+h]$. Since $\widehat{R}_h(t)/h$ and $R_h(t)$ have the same limit when $h \to 0^+$, namely

$$\lim_{h\to 0^+} \frac{1}{h}\widehat{R}_h(t) = \lim_{h\to 0^+} \frac{\log \Pi^{\mathcal{U}}(t+h) - \log \Pi^{\mathcal{U}}(t)}{h} = \frac{d \log \Pi^{\mathcal{U}}(t)}{dt} = r(t),$$

then $r(t)$ is also called **instantaneous log-return** of the asset.