Options and Mathematics: Lecture 2

November 4, 2020

Basic financial concepts

Historical volatility and mean of log return

- The **historical volatility** $\sigma > 0$ of an asset measures the amplitude of the time fluctuations of the asset price, thereby giving information on its level of uncertainty.
- The **historical mean of log return** $\alpha \in \mathbb{R}$ of the asset measures the average trend, up or down, of the asset price.

Remark: The adjective "historical" means that these parameters refer to the behavior of the stock price in some interval of time *in the past.*

Computing α of a stock

Let $[t_0, t]$ be some interval of time in the past, with t denoting possibly the present time, and let $T = t - t_0 > 0$ be the length of this interval.

Let us divide $[t_0, t]$ into n equally long periods, say

$$t_0 < t_1 < t_2 < \cdots < t_n = t, \quad t_i - t_{i-1} = h, \quad \text{for all } i = 1, \dots, n.$$

The set of points $\{t_0, t_1, \dots, t_n\}$ is called a **uniform partition** of the interval $[t_0, t]$.

Denote the log-return of the stock price in the interval $[t_{i-1}, t_i]$ as

$$\widehat{R}_i = \log S(t_i) - \log S(t_{i-1}) = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right), \quad i = 1, \dots, n.$$

The average of the log-returns is

$$\widehat{R}(t) = \frac{1}{n}\sum_{i=1}^{n} \widehat{R}_i = \frac{1}{n}\log\left(\frac{S(t)}{S(t_0)}\right).$$

The **T-historical mean of log-return** of the stock is obtained by "annualizing" the average of log-returns, i.e., by dividing $\widehat{R}(t)$ by the length h of the interval in which the log returns are computed:

$$\boxed{\alpha_T(t) = \frac{1}{nh}\log\left(\frac{S(t)}{S(t_0)}\right) = \frac{1}{T}\log\left(\frac{S(t)}{S(t_0)}\right)}$$

Remark: α_T does *not* depend on the partition being used to compute it!

Computing σ of a stock

The (corrected) sample variance of the log-returns is

$$\Delta(t) = \frac{1}{n-1}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2.$$

The **T-historical variance** of the stock is obtained by "annualizing" $\Delta(t)$, i.e.,

$$\sigma_T^2(t) = \frac{1}{h}\frac{1}{n-1}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2$$

The square root of the $T-$historical variance is the **T-historical volatility** of the stock:

$$\boxed{\sigma_T(t) = \frac{1}{\sqrt{h}}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2}}$$

Remark: σ_T depends on the partition being used to compute it.

Example: assume $t - t_0 = T = 20$ days, and let $t_1, \dots, t_{20}$ be the market closing times at these days. Let $h = 1$ day $= 1/252$ years. Then

$$\sigma_{20}(t) = \sqrt{252}\sqrt{\frac{1}{19}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2}$$

is called the 20days-historical volatility.

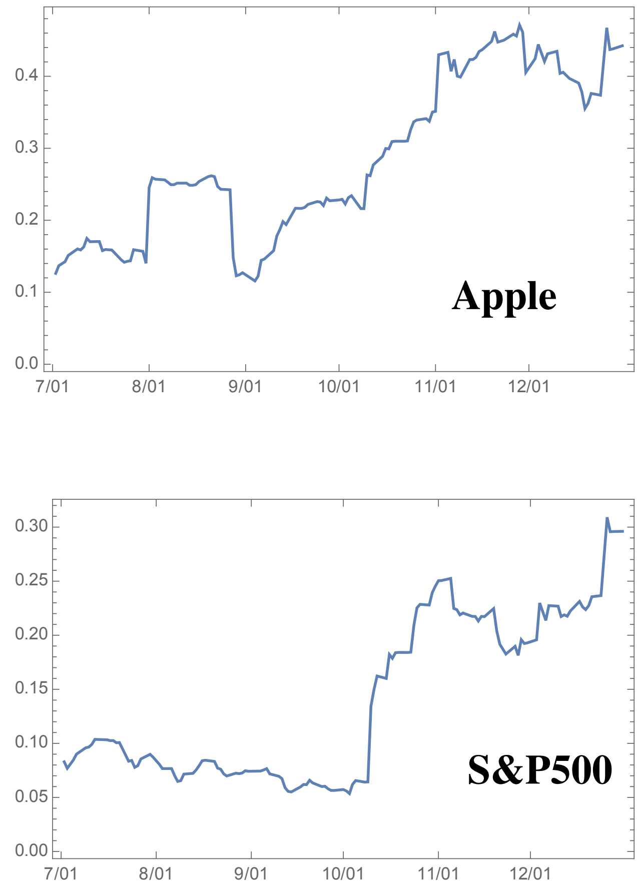

Figure 1: 20-days volatility of the Apple stock and the S&P500 index from July 1$^{\text{st}}$, 2018 until December 31$^{\text{st}}$, 2018.

Remark: S&P 500 is a **market index**. It measures the average value of 500 stocks traded in the New York stock exchange (NYSE) and NASDAQ markets

Remark: You can get today's value of σ_{20} for, say, the Apple stock by running the following command on Mathematica:

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FinancialData["AAPL", "Volatility20Day"]
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Options

Options are examples of **financial derivatives**, i.e., financial assets whose value depends (derives) on the value of one or more other financial assets, called underlying assets of the derivative.

We discuss here options on a single stock (**stock options**).

Call and Put options (also called vanilla options)

A **call option** is a contract that gives to its owner the right, but *not* the obligation, to buy the underlying stock for a given price K, which is fixed at the time when the contract is stipulated, and which is called **strike price** of the call.

If the buyer of the option can exercise this right only at some given time T in the future then the call option is called **European**, while if the option can be exercised at any time earlier than or equal to T, then the option is called **American**.

The time T is called **maturity time**, or **expiration date** of the call.

If the option to buy in the definition of a call is replaced by the option to sell, then the option is called a **put option**.

Pay-off of European calls/puts

The owner of the option should exercise the European call/put if (and only if) the **pay-off** at maturity is positive.

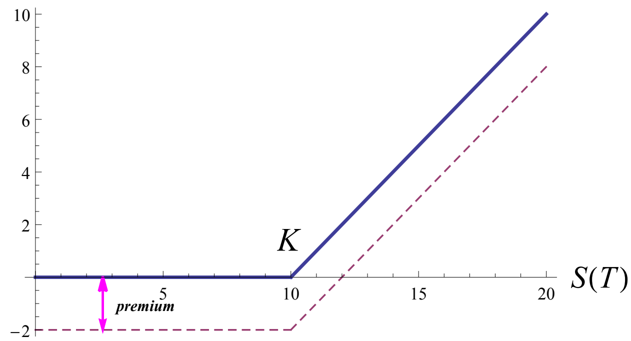
$$\boxed{Y_{\text{call}} = (S(T) - K)_+, \quad Y_{\text{put}} = (K - S(T))_+}$$

where $(x)_+ = \max(0, x)$.

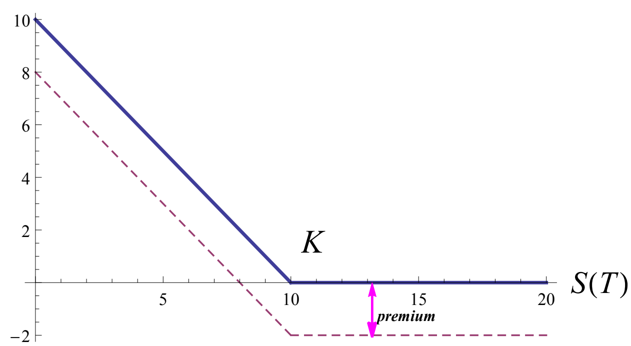
Terminology

- The European call (resp. put) with strike K is said to be **in the money** at time t if $S(t) > K$ (resp. $S(t) < K$).
- The call (resp. put) is said to be **out of the money** at time t if $S(t) < K$ (resp. $S(t) > K$). If $S(t) = K$, the (call or put) option is said to be **at the money** at time t.
- If the pay-off at maturity is positive, the option is said to expire in the money, otherwise it expires out of the money.

Stock options are *not* free! Let Π_0 be the price payed by the buyer of the option, also called **premium**. Then the return of the option will be positive if and only if the pay-off at maturity is larger than Π_0:



(a) Call option



(b) Put option

Figure 2: Pay-off function (continuous line) and return (dashed line) of some standard European derivatives.

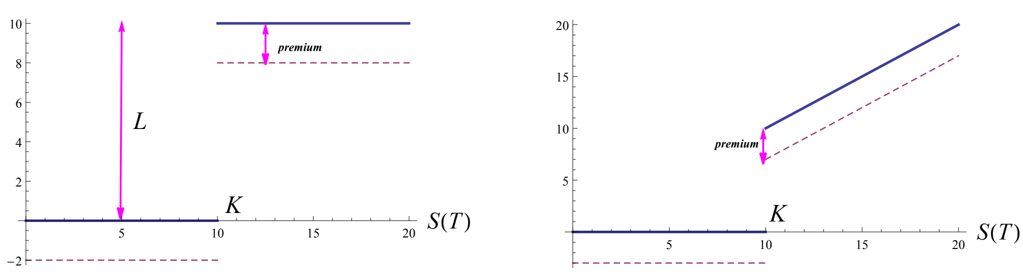
European Digital (or Binary) options

Denote by $H(x)$ the **Heaviside function**,

$$\boxed{H(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases}}$$

and let $K, L > 0$ be constants expressed in units of some currency (e.g., dollars).

- The European derivative with pay-off $Y = LH(S(T) - K)$ is called **cash-settled digital call option** with strike price K and **notional value** L; this option pays L if $S(T) > K$ and nothing otherwise.
- The **physically-settled digital call option** has the pay-off $Y = S(T)H(S(T) - K)$, which means that at maturity the buyer receives either the stock (when $S(T) > K$), or nothing.
- Similarly one defines the corresponding put options



(a) Digital call option (cash-settled)

(b) Digital call option (physically-settled)

Figure 3: Pay-off function (continuous line) and return (dashed line) of some standard European derivatives.

Asian options

The European stock options with pay-off

$$\boxed{Y_{\mathrm{AC}} = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+, \quad Y_{\mathrm{AP}} = \left(K - \frac{1}{T}\int_0^T S(t)\,dt\right)_+}$$

are called respectively the **Asian call** and **Asian put** with maturity T and strike K.

Note that for Asian options *the pay-off depends on the history of the stock price from today until maturity, and not only on the stock price at maturity (as for call/put/digital options)*

Terminology

- A European style derivative will be called **standard** if its pay-off depend only on the stock price at maturity, i.e., $Y = g(S(T))$, where $g(x)$ is called **pay-off function** (e.g., $g(x) = (x - K)_+$ for calls).
- A European derivative will be called **non-standard** if the pay-off depends on the stock price at different times, e.g., as in the Asian option or in the so called **Look-back call option**:

$$Y = \left(\max_{t\in[0,T]} S(t) - K\right)_+$$

Remark: The terminology standard/non-standard derivative is used in this course of easy reference. *It is not used in the financial world!*

Remark: All non-standard derivatives are traded OTC.

Standard American derivatives

In this course we will study also **standard American derivative**, which means that the pay-off to exercise at time $t \leq T$ is a function of the stock price at time t:

$$Y(t) = g(S(t)), \quad t \leq T$$

The quantity $Y(t)$ is also called **intrinsic value** of the American derivative

Examples:

- $Y(t) = (S(t) - K)_+$, i.e., $g(x) = (x - K)_+$ for the American call option
- $Y(t) = (K - S(t))_+$, i.e., $g(x) = (K - x)_+$ for the American put option

Remark: In most cases, the call/put options traded in the exchange market (e.g., the Chicago Board of Options Exchange) are of American style