Options and Mathematics: Lecture 7

November 12, 2020

The Binomial Model

Self-financing portfolio

A portfolio process in the binomial market is self-financing if purchasing more shares of one asset is possible only by selling shares of the other asset for an equivalent value (and not by infusing new cash into the portfolio), and, conversely, if any cash obtained by selling shares of one asset is immediately re-invested to buy shares of the other asset (and not withdrawn from the portfolio).

Let $(h_S(t-1), h_B(t-1))$ be the investor position on the stock and the risk-free asset during the time interval $(t-2, t-1]$.

Assume that the investor change the position on the assets at time $t-1$ and let $(h_S(t), h_B(t))$ be the new position in the interval $(t-1, t]$.

The value of the portfolio process at time $t-1$ is

$$V(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1),$$

while the value "immediately after" changing the position at time $t-1$ is

$$V'(t-1) = h_S(t)S(t-1) + h_B(t)B(t-1).$$

The difference $V'(t-1)-V(t-1)$, if not zero, corresponds to cash withdrawn or added to the portfolio as a result of the change in the position on the assets.

In a self-financing portfolio this difference must be zero. We thus must have $V'(t-1) = V(t-1)$, which leads to the following definition.

Definition A portfolio process $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ invested in a binomial market is said to be **self-financing** if

$$\boxed{h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1)}$$

holds for all $t \in \mathcal{I}$.

Exercise 2.4

Consider a 3-period binomial model with the following parameters:

$$u = \log\frac{5}{4}, \quad d = \log\frac{1}{2}, \quad r = \log\frac{3}{4}, \quad S(0) = B(0) = 64$$

and the portfolio process in this market given by

$h_S(1) = 1, \quad h_B(1) = -1,$
$h_S(2,u) = -1, \quad h_B(2,u) = 7/3, \quad h_S(2,d) = 7/2, \quad h_B(2,d) = -8/3,$
$h_S(3,u,u) = 1, \quad h_B(3,u,u) = -29/9, \quad h_S(3,d,d) = 49/4, \quad h_B(3,d,d) = -59/9,$
$h_S(3,u,d) = h_S(3,d,u) = -7/2, \quad h_B(3,u,d) = h_B(3,d,u) = 46/9.$

Show that this portfolio is self-financing. Compute the value of the portfolio process along any possible path and represent the result with a binomial tree.

Notation

We define the parameters q_u, q_d as

$$\boxed{q_u = q, \quad q_d = 1 - q, \quad \text{where} \quad q = \frac{e^r - e^d}{e^u - e^d}}$$

Note that (q_u, q_d) is the unique solution of the linear system

$$q_u + q_d = 1, \quad q_u e^u + q_d e^d = e^r.$$

Given a self-financing portfolio process, we denote

$$V^u(t) = h_S(t)S(t-1)e^u + h_B(t)B(t-1)e^r,$$

which is the value of the portfolio at time t assuming that the stock price goes up at time t, and

$$V^d(t) = h_S(t)S(t-1)e^d + h_B(t)B(t-1)e^r,$$

which is the value of the portfolio at time t, assuming that the stock price goes down at time t.

Note that

$$V^u(t) = V^u(t, x_1, \ldots, x_{t-1}) = V(t, x_1, \ldots, x_{t-1}, u),$$

and similarly for $V^d(t)$.

We now prove two identities satisfied by the value of self-financing portfolios.

Theorem 2.1 (recurrence formula for the value of self-financing portfolios)

The value $V(t)$ of a self-financing portfolio process $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ satisfies the following recurrence formula:

$$V(t) = e^{-r}[q_u V^u(t+1) + q_d V^d(t+1)], \quad \text{for } t = 0, \ldots, N-1.$$

Proof: Using the definition of $V^u(t), V^d(t)$, we have

$$\begin{aligned} e^{-r}[q_u V^u(t+1) + q_d V^d(t+1)] &= e^{-r}[q_u(h_S(t+1)S(t)e^u + h_B(t+1)B(t)e^r) \\ &\quad + q_d(h_S(t+1)S(t)e^d + h_B(t+1)B(t)e^r)] \\ &= e^{-r}[h_S(t+1)S(t)(q_u e^u + q_d e^d) + h_B(t+1)B(t)e^r(q_u + q_d)] \\ &= h_S(t+1)S(t) + h_B(t+1)B(t), \end{aligned}$$

By definition of self-financing portfolio, the last member of equals $V(t)$, and so the theorem is proved.

Theorem 2.5

Let $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ be a self-financing portfolio process with value $V(N, x)$ at time $t = N$ along the path x. The portfolio value $V(t)$ at earlier times satisfies

$$V(t) = e^{-r(N-t)} \sum_{(x_{t+1},\ldots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} V(N, x), \quad \text{for } t = 0, \ldots, N-1.$$

In particular, at time $t = 0$ we have

$$V(0) = e^{-rN} \sum_{x\in\{u,d\}^N} q_u^{N_u(x)} q_d^{N_d(x)} V(N, x).$$

Exercise 2.5

Show that the self-financing portfolio process in Exercise 2.4 satisfies the formulas in the previous theorems. HINT: Use the binomial tree of $V(t)$ derived in Exercise 2.4.