Options and Mathematics: Lecture 8

November 13, 2020

The Binomial Model

Portfolio generating a cash flow

When some cash is added or removed from the portfolio $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$, we say that the portfolio generates a cash flow.

Let $(h_S(t-1), h_B(t-1))$ be the investor position on the stock and the risk-free asset during the time interval $(t-2, t-1]$. The portfolio value at time $t-1$ is

$$V(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1)$$

Suppose that at the time $t-1$, the investor sells/buys shares of the two assets.

Let $(h_S(t), h_B(t))$ be the new position on the stock and the risk-free asset in the interval $(t-1, t]$.

Then the value of the portfolio process immediately after changing the position at time $t-1$ is given by

$$V'(t-1) = h_S(t)S(t-1) + h_B(t)B(t-1)$$

The cash flow $C(t)$ is defined as

$$V'(t-1) - V(t-1) = -C(t-1)$$

and corresponds to cash withdrawn (if $C(t-1) > 0$) or added (if $C(t-1) < 0$) to the portfolio as a result of changing the position on the assets

This leads to the following definition.

Definition 2.6

A portfolio process $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ is said to generate the **cash flow** $\{C(t), t = 0, \ldots, N-1\}$, if, for $t \in \mathcal{I}$,

$$\boxed{h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1) - C(t-1)}$$

or, equivalently, letting $\Delta f(t) = f(t) - f(t-1)$,

$$\Delta V(t) = h_S(t)\Delta S(t) + h_B(t)\Delta B(t) - C(t-1).$$

In particular: if $C(t-1) > 0$, then the cash is withdrawn from the portfolio, causing a decrease of its value

if $C(t-1) < 0$, then the cash is added to the portfolio, causing an increasing of its value.

Remarks

- As we assume $h_S(0) = h_S(1)$ and $h_B(0) = h_B(1)$, then $C(0) = 0$. Therefore the first time at which the investor can add/remove cash from the portfolio is after changing the position (instantaneously) at time $t = 1$, i.e., when passing from $(h_S(1), h_B(1))$ to $(h_S(2), h_B(2))$, generating the cash flow $C(1)$.

- The cash flow is not defined at time $t = N$, as the portfolio process has no value "immediately after" time N.

- Like portfolio positions and portfolio values, the cash flow generated by a portfolio process is also in general path dependen. As usual, we assume that the cash flow $C(t)$ depends only on the information available at, and no later than, time t. Equivalently, $C(t) = C(t, x_1, x_2, \ldots, x_t)$.

- The total cash flow generated by the portfolio along the path $x \in \{u, d\}^N$ is given by

$$C_{\text{tot}}(x) = \sum_{t=1}^{N-1} C(t, x_1, \ldots, x_t).$$

Exercise 2.6

Consider a 3-period binomial model with the following parameters:

$$u = \log \frac{5}{4}, \quad d = \log \frac{1}{2}, \quad r = \log \frac{3}{4}, \quad S(0) = B(0) = 64$$

and the portfolio process in this market given by

$$\begin{aligned}
&h_S(1) = 1, \quad h_B(1) = -1, \\
&h_S(2,u) = -2, \quad h_B(2,u) = 3, \quad h_S(2,d) = 2, \quad h_B(2,d) = -1, \\
&h_S(3,u,u) = 2, \quad h_B(3,u,u) = 2, \quad h_S(3,d,d) = -1, \quad h_B(3,d,d) = -2, \\
&h_S(3,u,d) = h_S(3,d,u) = 3, \quad h_B(3,u,d) = h_B(3,d,u) = 0.
\end{aligned}$$

Compute the cash flow generated by this portfolio process and express the result with a binomial tree.

ANSWER: $C(0) = 0$ (which is always true by definition of cash flow), $C(1,u) = 48$, $C(1,d) = -32$, $C(2,u,u) = -364$, $C(2,u,d) = -92$, $C(2,d,u) = -76$, $C(2,d,d) = 84$. Since $C(2,u,d) \neq C(2,d,u)$, the binomial tree of the cash flow generated by this portfolio process is not recombining.

Arbitrage portfolio

Next we prove that the binomial market does not admit self-financing arbitrage portfolio processes, provided the market parameters satisfy $d < r < u$ (no condition is required on the probability p).

Definition 2.7

A portfolio process $\{(h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ invested in a binomial market is called an **arbitrage portfolio process** if it is predictable and if its value $V(t)$ satisfies

1) $V(0) = 0$;
2) $V(N, x) \geq 0$, for all $x \in \{u, d\}^N$;
3) There exists $y \in \{u, d\}^N$ such that $V(N, y) > 0$.

Comments

- Condition 1) means that no initial wealth is required to set up the portfolio, i.e., the long and short positions on the two assets are perfectly balanced.
- Condition 2) means that the investor is sure not to loose money with this investment: regardless of the path followed by the stock price, the return of the portfolio is always non-negative.
- Condition 3) means that there is a strictly positive probability to make a profit, since along at least one path of the stock price the return of the portfolio is strictly positive.

Theorem 2.4

There exists a self-financing arbitrage portfolio in the binomial market if and only if $r \notin (d, u)$

Proof for $N = 1$

We have

$$h_S(0) = h_S(1) = h_S, \quad h_B(0) = h_B(1) = h_B,$$

i.e., the portfolio position in the 1-period model is constant (and thus predictable and self-financing) over the interval $[0, 1]$.

The value of the portfolio at time $t = 0$ is

$$V(0) = h_S S_0 + h_B B_0,$$

while at time $t = 1$ it is given by one of the following:

$$V(1) = V(1, u) = h_S S_0 e^u + h_B B_0 e^r,$$

if the stock price goes up at time $t = 1$, or

$$V(1) = V(1, d) = h_S S_0 e^d + h_B B_0 e^r,$$

if the stock price goes down at time $t = 1$.

Thus the portfolio is an arbitrage if $V(0) = 0$, i.e.,

$$h_S S_0 + h_B B_0 = 0,$$

if $V(1) \geq 0$, i.e.,

$$h_S S_0 e^u + h_B B_0 e^r \geq 0$$
$$h_S S_0 e^d + h_B B_0 e^r \geq 0$$

and if at least one of these inequalities is strict.

Now assume that (h_S, h_B) is an arbitrage portfolio.

Then $V(0) = 0$ implies $h_B B_0 = -h_S S_0$ and therefore the inequalities become

$$h_S S_0(e^u - e^r) \geq 0$$
$$h_S S_0(e^d - e^r) \geq 0$$

Since at least one of the inequalities must be strict, then $h_S \neq 0$.

If $h_S > 0$, then the first of the previous inequalities gives $u \geq r$, while the second gives $d \geq r$.

As $u > d$, the last two inequalities are equivalent to $r \leq d$.

Similarly, for $h_S < 0$ we obtain $u \leq r$ and $d \leq r$ which, again due to $u > d$, are equivalent to $r \geq u$.

We conclude that the existence of an arbitrage portfolio implies $r \leq d$ or $r \geq u$, that is $r \notin (d, u)$, which proves the "only if" part of the theorem.

To establish the "if" part we construct an arbitrage portfolio when $r \notin (d, u)$.

Assume $r \leq d$. Let us pick $h_S = 1$ and $h_B = -S_0/B_0$. Then $V(0) = 0$ and

$$h_S S_0 e^d + h_B B_0 e^r = S_0(e^d - e^r) \geq 0$$

Moreover, since $u > d$,

$$h_S S_0 e^u + h_B B_0 e^r = S_0(e^u - e^r) > S_0(e^d - e^r) \geq 0,$$

This shows that one can construct an arbitrage portfolio if $r \leq d$ and a similar argument can be used to find an arbitrage portfolio when $r \geq u$.

The proof of the theorem for the 1-period model is complete

Very important remark!

The condition $d < r < u$ for the absence of self-financing arbitrage portfolios in the binomial market is equivalent to

$$q_u \in (0, 1), \quad q_d = 1 - q_u \in (0, 1)$$

where we recall that

$$q_u = \frac{e^r - e^d}{e^u - e^d}$$

Hence a binomial market is (self-financing) arbitrage-free if and only if the pair (q_u, q_d) defines a probability vector, which is called **risk-neutral**, or **martingale** probability vector of the binomial market.

The reason for this terminology will be explained later in the course.