# Options and Mathematics: Lecture 9

November 17, 2020

# Binomial price of European derivatives

Consider a European derivative on the stock expiring at time T = N.

Recall that European derivatives can be exercised only at maturity.

The derivative will be called **standard** if its pay-off depends only on the price of the stock at maturity, i.e., Y = g(S(N)), for some function  $g : (0, \infty) \to \mathbb{R}$ , which is called the **pay-off function** of the derivative.

The derivative will be called **non-standard** if the pay-off is a (deterministic) function of the stock price at time t = N and at times earlier than maturity, i.e.,  $Y = g(S(0), \ldots, S(N))$ , where now  $g: (0, \infty)^{N+1} \to \mathbb{R}$ .

In both cases the pay-off depends on the path  $x = (x_1, \ldots, x_N) \in \{u, d\}^N$ followed by the stock price Assume that a European derivative is sold at time t < T for the price  $\Pi_Y(t)$ .

The first concern of the seller is to **hedge** the derivative, i.e., to invest the premium  $\Pi_Y(t)$  in such a way that the seller portfolio value at the expiration date is enough to pay-off the buyer of the derivative.

We assume that the seller invests the premium in the binomial market consisting of the underlying stock and the risk-free asset (**delta-hedging**).

## Definition 3.2

An **hedging** portfolio process for the European derivative with pay-off Y and maturity T = N is a predictable portfolio process  $\{(h_S(t), h_B(t))\}_{t \in \mathcal{I}}$ invested in the underlying stock and the risk-free asset such that its value V(t) satisfies V(N) = Y.

If  $V(t) = \Pi_Y(t)$  holds for all t = 0, ..., N, and not only at maturity, we say that  $\{h_S(t), h_B(t)\}_{t \in \mathcal{I}}$  is a **replicating** portfolio process for the given derivative.

The value V(t) of any self-financing hedging portfolio at time t is given by

$$V(t) = e^{-r(N-t)} \sum_{\substack{(x_{t+1},\dots,x_N)\in\{u,d\}^{N-t} \\ (x_{t+1},\dots,x_N)\in\{u,d\}^{N-t}}} q_{x_{t+1}}\cdots q_{x_N}V(N,x)$$
  
=  $e^{-r(N-t)} \sum_{\substack{(x_{t+1},\dots,x_N)\in\{u,d\}^{N-t}}} q_{x_{t+1}}\cdots q_{x_N}Y(x_1,\dots,x_N).$ 

#### Definition 3.3

The **binomial** (fair) price at time t = 0, ..., N - 1 of the European derivative with pay-off Y and maturity T = N is given by

$$\left(\Pi_Y(t) := e^{-r(N-t)} \sum_{(x_{t+1},\dots,x_N) \in \{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} Y(x_1,\dots,x_N)\right)$$

while  $\Pi_Y(N) := Y$ . In particular at time t = 0,

$$\left(\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x)\right)$$

where  $N_u(x)$  in the number of u's in x and  $N_d(x) = N - N_u(x)$  the number of d's.

#### **Remarks:**

- 1. The binomial price at time t of the European derivative equals the value required to open at time t a self-financing hedging portfolio process for the derivative. In particular, self-financing hedging portfolios of European derivatives in a binomial market are also replicating portfolios.
- 2. Note carefully that we have *not* proved yet that hedging self-financing portfolios exist. The existence of self-financing hedging portfolios is proved in later.

Note that

$$\Pi_Y(t) = e^{-r(N-t)} \sum_{(x_{t+1},\dots,x_N) \in \{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} Y(x_1,\dots,x_N) = \Pi_Y(t,x_1,\dots,x_t)$$

hence the binomial price of the derivative at time t depends only on the information available at time t and not on the uncertain future.

# Example

Recall that

$$S(N, x) = S_0 \exp(x_1 + \dots + x_N), \quad S(t, x_1, \dots, x_t) = S_0 \exp(x_1 + \dots + x_t)$$

hence

$$S(N, x) = S(t, x_1, \dots, x_t) \exp(x_{t+1} + \dots + x_N),$$

and therefore the binomial fair price for the standard European derivative with pay-off Y = g(S(N)) can be written as

$$\Pi_Y(t, x_1, \dots, x_t) = e^{-r(N-t)} \sum_{(x_{t+1}, \dots, x_N) \in \{u, d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} g(S(t, x_1, \dots, x_t) e^{x_{t+1} + \dots + x_N}).$$

This shows that the binomial price at time t of standard European derivatives is a deterministic function of S(t), namely

$$\left(\Pi_Y(t) = v_t(S(t))\right)$$

where

$$v_t(z) = e^{-r(N-t)} \sum_{(x_{t+1},\dots,x_N) \in \{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} g(z \exp(x_{t+1} + \dots + x_N))$$

is called the **pricing function** of the derivative (at time t).

In the particular case of the European call, respectively put, with strike K and maturity T = N, the binomial price at time t = 0, ..., N - 1 can be written in the form C(t, S(t), K, N), respectively P(t, S(t), K, N), where

$$C(t, S(t), K, T) = e^{-r(T-t)} \sum_{\substack{(x_{t+1}, \dots, x_T) \in \{u, d\}^{T-t}}} q_{x_{t+1}} \cdots q_{x_T} (S(t)e^{x_{t+1} + \dots + x_T} - K)_+,$$

$$P(t, S(t), K, T) = e^{-r(T-t)} \sum_{\substack{(x_{t+1}, \dots, x_T) \in \{u, d\}^{T-t}}} q_{x_{t+1}} \cdots q_{x_T} (K - S(t)e^{x_{t+1} + \dots + x_T})_+.$$

### **Remark:**

These explicit formulas can be used to give an alternative proof of the properties on European call/put options derived in the first week, see Theorem 3.1 in the lecture notes.

#### Recurrence formula for the binomial price

Let  $\Pi_Y^u(t)$  denote the binomial fair price of the European derivative at time t assuming that the stock price goes up at time t (i.e.,  $S(t) = S(t-1)e^u$ , or equivalently,  $x_t = u$ )

Note that

$$\Pi_Y^u(t) = \Pi_Y^u(t, x_1, \dots, x_{t-1}) = \Pi_Y(t, x_1, \dots, x_{t-1}, u).$$

Similarly define  $\Pi_Y^d(t)$ , with "up" replaced by "down".

By the proven recurrence formula for the value of self-financing portfolios we have the following important result.

The binomial price of European derivatives satisfies the recurrence formula

$$\Pi_Y(N) = Y$$

 $\left(\Pi_Y(t) = e^{-r} [q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)], \text{ for } t \in \{0, \dots, N-1\}\right)$ 

# Example: A standard European derivative

Consider the standard European derivative with pay-off  $Y = (\sqrt{S(2)} - 1)_+$ at maturity time T = 2.

Assume that the market parameters are given by

$$u = \log 2$$
,  $d = 0$ ,  $r = \log(4/3)$ ,  $p = 1/4$ .

Assume also  $S_0 = 1$ .

In this example we compute the possible paths for the binomial price  $\Pi_Y(t)$  of the derivative and the probability that the derivative expires in the money.

The stock price and the risk-free asset satisfy

$$S(t) = \begin{cases} S(t-1)e^u \\ S(t-1)e^d \end{cases}, \quad B(t) = B_0 e^{rt} \qquad t \in \{1,2\},$$

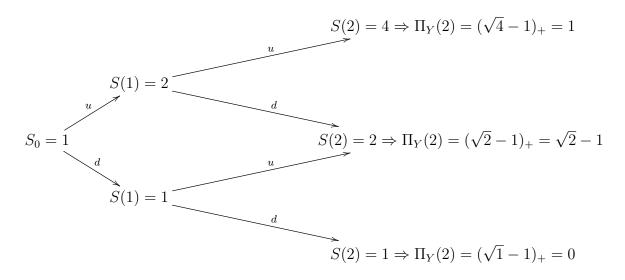
where

$$e^u = 2, \quad e^d = 1, \quad e^r = 4/3.$$

Hence

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1}{3}, \quad q_d = 1 - q_u = \frac{2}{3}.$$

Now, let us write the binomial tree of the stock price, including the possible values of the derivative at the expiration time T = 2 (where we use that  $\Pi_Y(2) = Y$ ):



Using the recurrence formula

$$\Pi_Y(t) = e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1))$$

we have, at time t = 1,

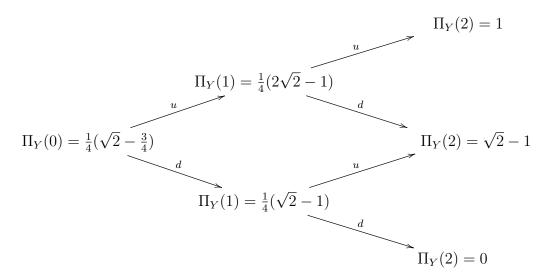
$$S(1) = S(1, u) = 2 \Rightarrow \Pi_Y(1) = \Pi_Y(1, u) = e^{-r} (q_u \Pi_Y^u(2, u) + q_d \Pi_Y^d(2, u))$$
  
=  $e^{-r} (q_u \Pi_Y(2, u, u) + q_d \Pi_Y(2, u, d))$   
=  $\frac{3}{4} (\frac{1}{3} \cdot 1 + \frac{2}{3}(\sqrt{2} - 1)) = \frac{1}{4}(2\sqrt{2} - 1)$ 

$$S(1) = S(1,d) = 1 \Rightarrow \Pi_Y(1) = \Pi_Y(1,d) = e^{-r}(q_u \Pi_Y^u(2,d) + q_d \Pi_Y^d(2,d))$$
  
=  $e^{-r}(q_u \Pi_Y(2,d,u) + q_d \Pi_Y(2,d,d))$   
=  $\frac{3}{4}(\frac{1}{3}(\sqrt{2}-1) + \frac{2}{3}\cdot 0) = \frac{1}{4}(\sqrt{2}-1)$ 

while at time t = 0 we have

$$\Pi_Y(0) = e^{-r} (q_u \Pi_Y^u(1) + q_d \Pi_Y^d(1))$$
  
=  $e^{-r} (q_u \Pi_Y(1, u) + q_d \Pi_Y(1, d))$   
=  $\frac{3}{4} (\frac{1}{3} \cdot \frac{1}{4} (2\sqrt{2} - 1) + \frac{2}{3} \cdot \frac{1}{4} (\sqrt{2} - 1)) = \frac{1}{4} (\sqrt{2} - \frac{3}{4}).$ 

Hence we have found the following diagram for the binomial price of the derivative



As to the probability that the derivative expires in the money, i.e.,  $\mathbb{P}(Y > 0)$ , we see from the above diagram that this happens along the paths (u, u), (u, d), (d, u), hence

$$\mathbb{P}(Y>0) = \mathbb{P}(S^{(u,u)}) + \mathbb{P}(S^{(u,d)}) + \mathbb{P}(S^{(d,u)}) = \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{7}{16},$$

which corresponds to 43,75%.

# Example: A non-standard European derivative

Consider a 3-period binomial market with the parameters  $e^u = \frac{4}{3}$ ,  $e^d = \frac{2}{3}$ ,  $p = \frac{3}{4}$ ,  $S_0 = 2$  and r = 0.

In this example we shall compute the binomial price at time t = 0 of the European derivative with pay-off

$$Y = \left(\frac{11}{9} - \min(S_0, S(1), S(2), S(3))\right)_+, \quad (z)_+ = \max(0, z),$$

and time of maturity T = 3.

This is an example of **lookback option**. We will also compute the probability that the derivative expires in the money and the probability that the return of a constant portfolio with a long position on this derivative be positive.

To compute the initial binomial price we use the formula

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x),$$

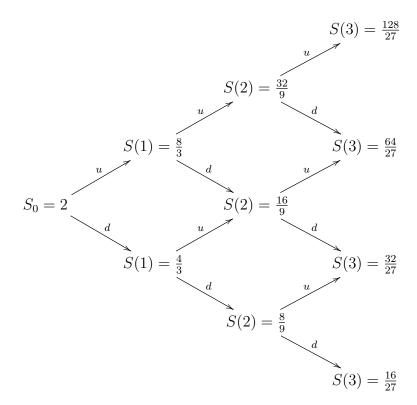
Here Y(x) denotes the pay-off as a function of the path of the stock price,  $N_u(x)$  is the number of times that the stock price goes up in the path x and  $N_d(x) = N - N_u(x)$  is the number of times that it goes down. In this example we have N = 3, r = 0 and

$$q_u = q_d = \frac{1}{2}.$$

So, it remains to compute the pay-off for all possible paths of the binomial stock price, where

$$Y = \left(\frac{11}{9} - \min(S_0, S(1), S(2), S(3))\right)_+, \quad (z)_+ = \max(0, z).$$

The binomial tree of the stock price is



From this we compute

$$Y(u, u, u) = \left(\frac{11}{9} - \min(2, 8/3, 32/9, 128/27)\right)_{+} = \left(\frac{11}{9} - 2\right)_{+} = \max(0, -\frac{7}{9}) = 0$$

$$Y(u, u, d) = \left(\frac{11}{9} - \min(2, 8/3, 32/9, 64/27)\right)_{+} = 0$$

$$Y(u, d, u) = \left(\frac{11}{9} - \min(2, 8/3, 16/9, 64/27)\right)_{+} = 1/27$$

$$Y(d, u, u) = \left(\frac{11}{9} - \min(2, 4/3, 16/9, 64/27)\right)_{+} = 0$$

$$Y(d, u, d) = \left(\frac{11}{9} - \min(2, 4/3, 16/9, 32/27)\right)_{+} = 1/27$$

$$Y(d, d, u) = \left(\frac{11}{9} - \min(2, 4/3, 16/9, 32/27)\right)_{+} = 1/27$$

$$Y(d, d, u) = \left(\frac{11}{9} - \min(2, 4/3, 8/9, 32/27)\right)_{+} = 1/3$$

$$Y(d, d, d) = \left(\frac{11}{9} - \min(2, 4/3, 8/9, 16/27)\right)_{+} = 1/27$$

Replacing in the formula for  $\Pi_Y(0)$  we obtain

$$\Pi_Y(0) = q_u(q_d)^2 Y(u, d, d) + (q_d)^2 q_u Y(d, u, d) + (q_d)^2 q_u Y(d, d, u) + (q_d)^3 Y(d, d, d),$$

the other terms being zero. Hence

$$\Pi_Y(0) = \frac{1}{8} \left( \frac{1}{27} + \frac{1}{27} + \frac{1}{3} + \frac{17}{27} \right) = \frac{7}{54}.$$

The probability that the derivative expires in the money is the probability that Y > 0. Hence we just sum the probabilities of the paths which lead to a positive pay-off:

$$\mathbb{P}(Y > 0) = \mathbb{P}(S^{(u,d,d)}) + \mathbb{P}(S^{(d,u,d)}) + \mathbb{P}(S^{(d,d,u)}) + \mathbb{P}(S^{(d,d,d)})$$
  
=  $p(1-p)^2 + (1-p)^2p + (1-p)^2p + (1-p)^3$   
=  $3(1-p)^2p + (1-p)^3 = 3\left(\frac{1}{4}\right)^2\frac{3}{4} + \left(\frac{1}{4}\right)^3 = \frac{5}{32} \approx 15,6\%$ 

Next consider a constant portfolio with a long position on the derivative. This means that the investor buys the derivative at time t = 0 and waits (without changing the portfolio) until the expiration time t = 3. The return will be positive (i.e., the buyer makes a profit) if and only if  $\Pi_Y(3) > \Pi_Y(0)$ . But  $\Pi_Y(3) = Y$ , which, according to the computations above, is greater than  $\Pi_Y(0) = 7/54$  only when the binomial stock price follows one of the paths (d, d, u) or (d, d, d). Hence

$$\mathbb{P}(R>0) = \mathbb{P}(S^{(d,d,u)}) + \mathbb{P}(S^{(d,d,d)}) = (1-p)^2 p + (1-p)^3 = (1-p)^2 = \frac{1}{16} \approx 6,2\%$$