Options and Mathematics: Lecture 9

November 17, 2020

Binomial price of European derivatives

Consider a European derivative on the stock expiring at time $T = N$.

Recall that European derivatives can be exercised only at maturity.

The derivative will be called **standard** if its pay-off depends only on the price of the stock at maturity, i.e., $Y = g(S(N))$, for some function $g : (0, \infty) \to \mathbb{R}$, which is called the **pay-off function** of the derivative.

The derivative will be called **non-standard** if the pay-off is a (deterministic) function of the stock price at time $t = N$ and at times earlier than maturity, i.e., $Y = g(S(0), \dots, S(N))$, where now $g : (0, \infty)^{N+1} \to \mathbb{R}$.

In both cases the pay-off depends on the path $x = (x_1, \dots, x_N) \in \{u, d\}^N$ followed by the stock price

Assume that a European derivative is sold at time $t < T$ for the price $\Pi_Y(t)$.

The first concern of the seller is to **hedge** the derivative, i.e., to invest the premium $\Pi_Y(t)$ in such a way that the seller portfolio value at the expiration date is enough to pay-off the buyer of the derivative.

We assume that the seller invests the premium in the binomial market consisting of the underlying stock and the risk-free asset (**delta-hedging**).

Definition 3.2

An **hedging** portfolio process for the European derivative with pay-off Y and maturity $T = N$ is a predictable portfolio process $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ invested in the underlying stock and the risk-free asset such that its value $V(t)$ satisfies $V(N) = Y$.

If $V(t) = \Pi_Y(t)$ holds for all $t = 0, \ldots, N$, and not only at maturity, we say that $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ is a **replicating** portfolio process for the given derivative.

The value $V(t)$ of *any* self-financing hedging portfolio at time t is given by

$$\begin{aligned}
V(t) &= e^{-r(N-t)} \sum_{(x_{t+1},\ldots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} V(N, x) \\
&= e^{-r(N-t)} \sum_{(x_{t+1},\ldots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} Y(x_1, \ldots, x_N).
\end{aligned}$$

Definition 3.3

The **binomial** (**fair**) **price** at time $t = 0, \dots, N-1$ of the European derivative with pay-off Y and maturity $T = N$ is given by

$$\boxed{\Pi_Y(t) := e^{-r(N-t)} \sum_{(x_{t+1},\dots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} Y(x_1,\dots,x_N)}$$

while $\Pi_Y(N) := Y$. In particular at time $t = 0$,

$$\boxed{\Pi_Y(0) = e^{-rN} \sum_{x\in\{u,d\}^N} (q_u)^{N_u(x)}(q_d)^{N_d(x)} Y(x)}$$

where $N_u(x)$ in the number of u's in x and $N_d(x) = N - N_u(x)$ the number of d's.

Remarks:

1. The binomial price at time t of the European derivative equals the value required to open at time t a self-financing hedging portfolio process for the derivative. In particular, self-financing hedging portfolios of European derivatives in a binomial market are also replicating portfolios.

2. Note carefully that we have *not* proved yet that hedging self-financing portfolios exist. The existence of self-financing hedging portfolios is proved in later.

Note that

$$\Pi_Y(t) = e^{-r(N-t)} \sum_{(x_{t+1},\ldots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} Y(x_1, \ldots, x_N) = \Pi_Y(t, x_1, \ldots, x_t)$$

hence the binomial price of the derivative at time t depends only on the information available at time $t$and not on the uncertain future.

Example

Recall that

$$S(N, x) = S_0 \exp(x_1 + \cdots + x_N), \quad S(t, x_1, \ldots, x_t) = S_0 \exp(x_1 + \cdots + x_t)$$

hence

$$S(N, x) = S(t, x_1, \ldots, x_t) \exp(x_{t+1} + \cdots + x_N),$$

and therefore the binomial fair price for the standard European derivative with pay-off $Y = g(S(N))$ can be written as

$$\Pi_Y(t, x_1, \ldots, x_t) = e^{-r(N-t)} \sum_{(x_{t+1},\ldots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} g(S(t, x_1, \ldots, x_t) e^{x_{t+1}+\cdots+x_N}).$$

This shows that the binomial price at time t of standard European derivatives is a deterministic function of $S(t)$, namely

$$\boxed{\Pi_Y(t) = v_t(S(t))}$$

where

$$v_t(z) = e^{-r(N-t)} \sum_{(x_{t+1},\ldots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} g(z \exp(x_{t+1} + \cdots + x_N))$$

is called the **pricing function** of the derivative (at time t).

In the particular case of the European call, respectively put, with strike K and maturity $T = N$, the binomial price at time $t = 0, \ldots, N-1$ can be written in the form $C(t, S(t), K, N)$, respectively $P(t, S(t), K, N)$, where

$$C(t, S(t), K, T) = e^{-r(T-t)} \sum_{(x_{t+1},\ldots,x_T)\in\{u,d\}^{T-t}} q_{x_{t+1}} \cdots q_{x_T} (S(t)e^{x_{t+1}+\cdots+x_T} - K)_+,$$

$$P(t, S(t), K, T) = e^{-r(T-t)} \sum_{(x_{t+1},\ldots,x_T)\in\{u,d\}^{T-t}} q_{x_{t+1}} \cdots q_{x_T} (K - S(t)e^{x_{t+1}+\cdots+x_T})_+.$$

Remark:

These explicit formulas can be used to give an alternative proof of the properties on European call/put options derived in the first week, see Theorem 3.1 in the lecture notes.

Recurrence formula for the binomial price

Let $\Pi_Y^u(t)$ denote the binomial fair price of the European derivative at time t assuming that the stock price goes up at time t (i.e., $S(t) = S(t-1)e^u$, or equivalently, $x_t = u$)

Note that

$$\Pi_Y^u(t) = \Pi_Y^u(t, x_1, \ldots, x_{t-1}) = \Pi_Y(t, x_1, \ldots, x_{t-1}, u).$$

Similarly define $\Pi_Y^d(t)$, with "up" replaced by "down".

By the proven recurrence formula for the value of self-financing portfolios we have the following important result.

The binomial price of European derivatives satisfies the recurrence formula

$$\boxed{\Pi_Y(N) = Y}$$

$$\boxed{\Pi_Y(t) = e^{-r}[q_u\Pi_Y^u(t+1) + q_d\Pi_Y^d(t+1)], \text{ for } t \in \{0, \ldots, N-1\}}$$

Example: A standard European derivative

Consider the standard European derivative with pay-off $Y = (\sqrt{S(2)} - 1)_+$ at maturity time $T = 2$.

Assume that the market parameters are given by

$$u = \log 2, \quad d = 0, \quad r = \log(4/3), \quad p = 1/4.$$

Assume also $S_0 = 1$.

In this example we compute the possible paths for the binomial price $\Pi_Y(t)$ of the derivative and the probability that the derivative expires in the money.

The stock price and the risk-free asset satisfy

$$S(t) = \begin{cases} S(t-1)e^u \\ S(t-1)e^d \end{cases}, \quad B(t) = B_0 e^{rt} \qquad t \in \{1, 2\},$$

where

$$e^u = 2, \quad e^d = 1, \quad e^r = 4/3.$$

Hence

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1}{3}, \quad q_d = 1 - q_u = \frac{2}{3}.$$

Now, let us write the binomial tree of the stock price, including the possible values of the derivative at the expiration time $T = 2$ (where we use that $\Pi_Y(2) = Y$):

$$
\begin{tikzcd}
 & & S(2) = 4 \Rightarrow \Pi_Y(2) = (\sqrt{4} - 1)_+ = 1 \\
 & S(1) = 2 \arrow[ur, "u"] \arrow[dr, "d"] & \\
S_0 = 1 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2) = 2 \Rightarrow \Pi_Y(2) = (\sqrt{2} - 1)_+ = \sqrt{2} - 1 \\
 & S(1) = 1 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & S(2) = 1 \Rightarrow \Pi_Y(2) = (\sqrt{1} - 1)_+ = 0
\end{tikzcd}
$$

Using the recurrence formula

$$\Pi_Y(t) = e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1))$$

we have, at time $t = 1$,

$$
\begin{aligned}
S(1) = S(1,u) = 2 \Rightarrow \Pi_Y(1) &= \Pi_Y(1,u) = e^{-r}(q_u \Pi_Y^u(2,u) + q_d \Pi_Y^d(2,u)) \\
&= e^{-r}(q_u \Pi_Y(2,u,u) + q_d \Pi_Y(2,u,d)) \\
&= \frac{3}{4}(\frac{1}{3} \cdot 1 + \frac{2}{3}(\sqrt{2} - 1)) = \frac{1}{4}(2\sqrt{2} - 1)
\end{aligned}
$$

$$
\begin{aligned}
S(1) = S(1,d) = 1 \Rightarrow \Pi_Y(1) &= \Pi_Y(1,d) = e^{-r}(q_u \Pi_Y^u(2,d) + q_d \Pi_Y^d(2,d)) \\
&= e^{-r}(q_u \Pi_Y(2,d,u) + q_d \Pi_Y(2,d,d)) \\
&= \frac{3}{4}(\frac{1}{3}(\sqrt{2} - 1) + \frac{2}{3} \cdot 0) = \frac{1}{4}(\sqrt{2} - 1)
\end{aligned}
$$

while at time $t = 0$ we have

$$
\begin{aligned}
\Pi_Y(0) &= e^{-r}(q_u \Pi_Y^u(1) + q_d \Pi_Y^d(1)) \\
&= e^{-r}(q_u \Pi_Y(1,u) + q_d \Pi_Y(1,d)) \\
&= \frac{3}{4}(\frac{1}{3} \cdot \frac{1}{4}(2\sqrt{2} - 1) + \frac{2}{3} \cdot \frac{1}{4}(\sqrt{2} - 1)) = \frac{1}{4}(\sqrt{2} - \frac{3}{4}).
\end{aligned}
$$

Hence we have found the following diagram for the binomial price of the derivative

$$
\begin{tikzcd}
 & & \Pi_Y(2) = 1 \\
 & \Pi_Y(1) = \frac{1}{4}(2\sqrt{2} - 1) \arrow[ur, "u"] \arrow[dr, "d"] & \\
\Pi_Y(0) = \frac{1}{4}(\sqrt{2} - \frac{3}{4}) \arrow[ur, "u"] \arrow[dr, "d"] & & \Pi_Y(2) = \sqrt{2} - 1 \\
 & \Pi_Y(1) = \frac{1}{4}(\sqrt{2} - 1) \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \Pi_Y(2) = 0
\end{tikzcd}
$$

As to the probability that the derivative expires in the money, i.e., $\mathbb{P}(Y > 0)$, we see from the above diagram that this happens along the paths (u,u), (u,d), (d,u), hence

$$
\mathbb{P}(Y > 0) = \mathbb{P}(S^{(u,u)}) + \mathbb{P}(S^{(u,d)}) + \mathbb{P}(S^{(d,u)}) = \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{7}{16},
$$

which corresponds to $43,75\%$.

Example: A non-standard European derivative

Consider a 3-period binomial market with the parameters $e^u = \frac{4}{3}$, $e^d = \frac{2}{3}$, $p = \frac{3}{4}$, $S_0 = 2$ and $r = 0$.

In this example we shall compute the binomial price at time $t = 0$ of the European derivative with pay-off

$$Y = \left(\frac{11}{9} - \min(S_0, S(1), S(2), S(3))\right)_+, \quad (z)_+ = \max(0, z),$$

and time of maturity $T = 3$.

This is an example of **lookback option**. We will also compute the probability that the derivative expires in the money and the probability that the return of a constant portfolio with a long position on this derivative be positive.

To compute the initial binomial price we use the formula

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x),$$

Here $Y(x)$ denotes the pay-off as a function of the path of the stock price, $N_u(x)$ is the number of times that the stock price goes up in the path x and $N_d(x) = N - N_u(x)$ is the number of times that it goes down. In this example we have $N = 3$, $r = 0$ and

$$q_u = q_d = \frac{1}{2}.$$

So, it remains to compute the pay-off for all possible paths of the binomial stock price, where

$$Y = \left(\frac{11}{9} - \min(S_0, S(1), S(2), S(3))\right)_+, \quad (z)_+ = \max(0, z).$$

The binomial tree of the stock price is

$$
\begin{tikzcd}
 & & & S(3) = \frac{128}{27} \\
 & & S(2) = \frac{32}{9} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1) = \frac{8}{3} \arrow[ur, "u"] \arrow[dr, "d"] & & S(3) = \frac{64}{27} \\
S_0 = 2 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2) = \frac{16}{9} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1) = \frac{4}{3} \arrow[ur, "u"] \arrow[dr, "d"] & & S(3) = \frac{32}{27} \\
 & & S(2) = \frac{8}{9} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & S(3) = \frac{16}{27}
\end{tikzcd}
$$

From this we compute

$$Y(u,u,u) = \left(\frac{11}{9} - \min(2, 8/3, 32/9, 128/27)\right)_+ = \left(\frac{11}{9} - 2\right)_+ = \max(0, -\frac{7}{9}) = 0$$
$$Y(u,u,d) = \left(\frac{11}{9} - \min(2, 8/3, 32/9, 64/27)\right)_+ = 0$$
$$Y(u,d,u) = \left(\frac{11}{9} - \min(2, 8/3, 16/9, 64/27)\right)_+ = 0$$
$$Y(u,d,d) = \left(\frac{11}{9} - \min(2, 8/3, 16/9, 32/27)\right)_+ = 1/27$$
$$Y(d,u,u) = \left(\frac{11}{9} - \min(2, 4/3, 16/9, 64/27)\right)_+ = 0$$
$$Y(d,u,d) = \left(\frac{11}{9} - \min(2, 4/3, 16/9, 32/27)\right)_+ = 1/27$$
$$Y(d,d,u) = \left(\frac{11}{9} - \min(2, 4/3, 8/9, 32/27)\right)_+ = 1/3$$
$$Y(d,d,d) = \left(\frac{11}{9} - \min(2, 4/3, 8/9, 16/27)\right)_+ = 17/27$$

Replacing in the formula for $\Pi_Y(0)$ we obtain

$$\Pi_Y(0) = q_u(q_d)^2Y(u,d,d)+(q_d)^2q_uY(d,u,d)+(q_d)^2q_uY(d,d,u)+(q_d)^3Y(d,d,d),$$

the other terms being zero. Hence

$$\Pi_Y(0) = \frac{1}{8}\Big(\frac{1}{27}+\frac{1}{27}+\frac{1}{3}+\frac{17}{27}\Big) = \frac{7}{54}.$$

The probability that the derivative expires in the money is the probability that $Y > 0$. Hence we just sum the probabilities of the paths which lead to a positive pay-off:

$$\begin{aligned}
\mathbb{P}(Y > 0) &= \mathbb{P}(S^{(u,d,d)}) + \mathbb{P}(S^{(d,u,d)}) + \mathbb{P}(S^{(d,d,u)}) + \mathbb{P}(S^{(d,d,d)}) \\
&= p(1-p)^2 + (1-p)^2p + (1-p)^2p + (1-p)^3 \\
&= 3(1-p)^2p + (1-p)^3 = 3\left(\frac{1}{4}\right)^2\frac{3}{4} + \left(\frac{1}{4}\right)^3 = \frac{5}{32} \approx 15,6\%
\end{aligned}$$

Next consider a constant portfolio with a long position on the derivative. This means that the investor buys the derivative at time $t = 0$ and waits (without changing the portfolio) until the expiration time $t = 3$. The return will be positive (i.e., the buyer makes a profit) if and only if $\Pi_Y(3) > \Pi_Y(0)$. But $\Pi_Y(3) = Y$, which, according to the computations above, is greater than $\Pi_Y(0) = 7/54$ only when the binomial stock price follows one of the paths (d,d,u) or (d,d,d). Hence

$$\mathbb{P}(R > 0) = \mathbb{P}(S^{(d,d,u)})+\mathbb{P}(S^{(d,d,d)}) = (1-p)^2p+(1-p)^3 = (1-p)^2 = \frac{1}{16} \approx 6,2\%$$