# Options and Mathematics: Lecture 10

November 18, 2020

## Replicating portfolio of European derivatives

Next we treat the important problem of building a self-financing replicating portfolio process for European derivatives.

**Theorem 3.3**

Consider the European derivative with pay-off $Y$ at the time of maturity $T = N$. Then the portfolio process given by

$$\boxed{h_S(t) = \frac{1}{S(t-1)} \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d}} \quad t \in \mathcal{I}$$

$$\boxed{h_B(t) = \frac{e^{-r}}{B(t-1)} \frac{e^u \Pi_Y^d(t) - e^d \Pi_Y^u(t)}{e^u - e^d}} \quad t \in \mathcal{I}$$

, is a self-financing replicating portfolio process.

*Proof.* We first show that the given portfolio replicates the derivative. We have

$$V(t) = h_S(t)S(t) + h_B(t)B(t) = \frac{S(t)}{S(t-1)} \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d} + \frac{e^{-r}B(t)}{B(t-1)} \frac{e^u \Pi_Y^d(t) - e^d \Pi_Y^u(t)}{e^u - e^d}.$$

Note that $e^{-r}B(t)/B(t-1) = 1$, while $S(t)/S(t-1)$ is either $e^u$ or $e^d$.

By straightforward calculations we obtain

$$\begin{aligned} V^u(t) &= h_S(t)S(t-1)e^u + h_B(t)B(t) \\ &= e^u \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d} + \frac{e^u \Pi_Y^d(t) - e^d \Pi_Y^u(t)}{e^u - e^d} = \Pi_Y^u(t), \end{aligned}$$

and similarly $V^d(t) = \Pi_Y^d(t)$. Thus $V(t) = \Pi_Y(t)$, for all $t \in \mathcal{I}$, i.e., the portfolio process is replicating, and therefore also hedging, the derivative.

As to the self-financing property, we have

$$\begin{aligned} h_S(t)S(t-1) + h_B(t)B(t-1) &= \frac{\Pi_Y^u(t)(1 - e^{d-r}) + \Pi_Y^d(t)(e^{u-r} - 1)}{e^u - e^d} \\ &= e^{-r}(q_u \Pi_Y^u(t) + q_d \Pi_Y^d(t)) = \Pi_Y(t-1), \end{aligned}$$

where we used the definition of $q_u, q_d$, as well as the recurrence formula for $\Pi_Y(t)$.

By the already proven fact that $V(t) = \Pi_Y(t)$, for all $t \in \mathcal{I}$, we have

$$h_S(t)S(t-1) + h_B(t)B(t-1) = V(t-1),$$

which proves the self-financing property.

Finally we show that the portfolio is predictable. Assume first that the European derivative is standard, i.e., $Y = g(S(N))$. Then $\Pi_Y(t) = v_t(S(t))$, and therefore

$$\Pi_Y^u(t) = v_t(S(t-1)e^u), \quad \Pi_Y^d(t) = v_t(S(t-1)e^d),$$

i.e., $\Pi_Y^u(t)$ and $\Pi_Y^d(t)$ are deterministic functions of $S(t-1)$. It follows that $h_S(t), h_B(t)$ are also deterministic functions of $S(t-1)$, and so this portfolio process is predictable.

In the case of non-standard derivatives we have similarly that $\Pi_Y^u(t)$ and $\Pi_Y^d(t)$ are deterministic functions of $S(1), \ldots, S(t-1)$, which again implies that the portfolio is predictable. $\square$

**Example.**

Let us compute the position on the stock in the hedging portfolio for the example of standard derivative considered before.

When the stock price goes up in the first step we have $S(1) = S(1,u) = 2$ and $\Pi_Y^u(2) = \Pi_Y^u(2,u) = \Pi_Y(2,u,u) = 1$, $\Pi_Y^d(2) = \Pi_Y^d(2,u) = \Pi_Y(2,u,d) = \sqrt{2}-1$, hence

$$h_S(2,u) = \frac{1}{S(1,u)} \frac{\Pi_Y^u(2,u) - \Pi_Y^d(2,u)}{e^u - e^d} = \frac{1-(\sqrt{2}-1)}{2-1} = 2-\sqrt{2} > 0 \quad \text{(long position)}.$$

When $S(1) = S(1,d) = 1$ we have $\Pi_Y^u(2) = \Pi_Y^u(2,d) = \Pi_Y(2,d,u) = \sqrt{2}-1$ and $\Pi_Y^d(2) = \Pi_Y^d(2,d) = \Pi_Y(2,d,d) = 0$,

hence

$$h_S(2,d) = \frac{1}{S(1,d)} \frac{\Pi_Y^u(2,d) - \Pi_Y^d(2,d)}{e^u - e^d} = \sqrt{2}-1 > 0 \quad \text{(long position)}.$$

Recall that $h_S(2)$ is the position in the stock in the interval $(1,2]$. In the interval $[0,1]$ we have

$$h_S(1) = \frac{1}{S(0)} \frac{\Pi_Y^u(1) - \Pi_Y^d(1)}{e^u - e^d} = \frac{1}{2} \frac{\frac{1}{4}(2\sqrt{2}-1) - \frac{1}{4}(\sqrt{2}-1)}{2-1} = \frac{\sqrt{2}}{4} > 0 \quad \text{(long position)}.$$

The result can be expressed in a binomial tree as follows:

$$\begin{tikzcd}
 & h_S(2) = 2 - \sqrt{2} \\
h_S(1) = \frac{\sqrt{2}}{4} \arrow[ur, "u"] \arrow[dr, "d"'] & \\
 & h_S(2) = \sqrt{2} - 1
\end{tikzcd}$$

The position on the risk-free asset can be computed likewise using the formula for $h_B(t)$ in the theorem.

**Exercise 3.11**

Consider a 3-period binomial market with the following parameters:

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1 \quad p = \frac{1}{2}.$$

Assume $S_0 = \frac{64}{25}$.

Consider the European derivative expiring at time $T = 3$ and with pay-off

$$Y = S(3)H(S(3) - 1),$$

where $H$ is the Heaviside function (this is an example of **physically-settled digital option**).

Compute the possible paths of the derivative price and for each of them give the number of shares of the underlying stock in the self-financing hedging portfolio process.