Options and Mathematics: Lecture 11

November 19, 2020

Exercises

Exercise 3.3

A **compound option** is an option whose underlying is another option.

For instance, given $T_2 > T_1 > 0$ and $K_1, K_2 > 0$, a **call on a put** with maturity T_1 and strike K_1 is a contract that gives to its owner the right to buy at time T_1 for the price K_1 the put option on the stock with maturity T_2 and strike K_2.

Let $S(t)$ be the price of the underlying stock of the put option. Assume that $S(t)$ follows a 2-period binomial model with parameters

$$e^u = \frac{7}{4}, \quad e^d = \frac{1}{2}, \quad e^r = \frac{9}{8}, \quad p = \frac{1}{4}, \quad S(0) = 16.$$

Assume further that $T_2 = 2$, $T_1 = 1$, $K_1 = \frac{23}{9}$, $K_2 = 12$.

Compute the initial price of the call on the put. Compute also the probability of positive return for the owner of the call on the put.

Exercise 3.23

Compute the self-financing hedging portfolio for the compound option in the previous exercise. Assume $B(0) = 1$.

Exercise 3.6

A **barrier option** is an option that expires worthless as soon as the stock price crosses a specified level (the barrier of the option

For example, consider a 3-period binomial market with parameters

$$e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad p = \frac{3}{4}, \quad S_0 = 2, \quad and r = 0$$

and the barrier call option with strike $K = 2$ and barrier $B = 3$.

This means that the derivative expires worthless if $S(3) \leq 2$ or if $S(t) > 3$ at some time $t \in \{1, 2, 3\}$.

Compute the binomial price $\Pi_Y(0)$ of this barrier option at time $t = 0$ and the probability that it expires in the money.

ANSWER: $\Pi_Y(0) = 5/54$. $\mathbb{P}(Y > 0) = 28,125\%$.

Exercise 3.7

Let $N = 2$ and

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1, \quad S_0 = \frac{64}{25}, \quad p \in (0,1).$$

Consider a European derivative with maturity $T = 2$ and pay-off

$$Y = \Big(\frac{1}{3}\sum_{i=0}^{2} S(i) - 2\Big)_+,$$

which is an example of **Asian** call option. Compute the price of the derivative at times $t = 0, 1, 2$.

ANSWER: $\Pi_Y(0) = 148/225$, $\Pi_Y(1,u) = 74/75$, $\Pi_Y(2,u,u) = 94/75$, $\Pi_Y(2,u,d) = 34/75$, $\Pi_Y(1,d) = \Pi_Y(2,d,u) - \Pi_Y(2,d,d) = 0$.

Exercise 3.8

The Asian call, resp. put, with strike K in a N-period binomial market is the non-standard European derivative with pay-off

$$Y_{\mathrm{AC}}(x) = \left(\frac{1}{N+1}\sum_{t=0}^{N} S(t) - K\right)_+, \quad \text{resp.} \quad Y_{\mathrm{AP}}(x) = \left(K - \frac{1}{N+1}\sum_{t=0}^{N} S(t)\right)_+.$$

Derive the following put-call parity satisfied by the binomial price at time $t = 0$ of the Asian option:

$$\Pi_{\mathrm{AC}}(0) - \Pi_{\mathrm{AP}}(0) = e^{-rN}\Big[\frac{S(0)}{N+1}\frac{e^{r(N+1)} - 1}{e^r - 1} - K\Big].$$

HINT: For $\alpha \neq 1$, $\sum_{k=0}^{N} \alpha^k = \frac{1-\alpha^{N+1}}{1-\alpha}$.